§1.4 Exact ODEs. Integrating Factors

Recall how for a fn $y = f(x)$ of one variable

$$dy = f'(x)\,dx = \left(\frac{dy}{dx}\right) dx$$

measures the infinitesimal change dy in y if we make an infinitesimal change dx in x.

For a fn $u = u(x,y)$ of two variables,

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

likewise measures the infinitesimal change du in u if we make infinitesimal changes dx, dy in x, y resp.

If we have $u(x,y) = c$ where c is a const, then clearly $du = 0$, ie

$$\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0$$

e.g. If $u = x + x^2y^3 = c$, then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

ie $$(1+2xy^3)dx + 3x^2y^2 dy = 0$$

$$\Rightarrow \quad 3x^2y^2 dy = -(1+2xy^3)dx$$

or $$y' = \frac{dy}{dx} = -\frac{(1+2xy^3)}{3x^2y^2}$$

which gives us an ODE which we can try to solve.

Leads to a powerful method for solving ODEs.

A first order ODE of the type

$$M(x,y) + N(x,y)y' = 0$$

can be written as (use $dy = y'dx$)

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

We say the ODE is <u>exact</u> if the <u>differential form</u> $M(x,y)dx + N(x,y)dy = 0$ is exact, ie we can find u st.

$$M(x,y)dx + N(x,y)dy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du.$$

Then $M(x,y)dx + N(x,y)\,dy = 0$

$$\Rightarrow \quad \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

ie $\quad du = 0$

and so on integrating $u(x,y) = C$.

This gives us an <u>implicit solution</u> where we may or may not be able to get y as a fn of x.

So we want to find u s.t.

$$\frac{\partial y}{\partial x} = M \quad , \quad \frac{\partial u}{\partial y} = N .$$

Suppose now that M, N are cts and have cts. first order partial derivs in a region in the x-y plane whose boundary is a closed curve without self-intersections.

Then by partial diff.

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad , \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} .$$

By our continuity assumptions these mixed partial derivs. will be equal.

Thus

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

We have shown this condition is <u>necessary</u>

i.e. if u exists, then it must be satisfied

It turns out that it is also <u>suffizient</u>

i.e. if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then u exists.

In practice, can get u by integrating.

e.g. Integrate $\frac{\partial u}{\partial x} = M$

to get $u = \int M\,dx + k(y)$

where $k(y)$ is some fn of y to be determined.

To find $k(y)$ diff wrt y and use $\frac{\partial u}{\partial y} = N$ to get $\frac{dk}{dy}$. Then integrate $\frac{dk}{dy}$ wrt y to get $k(y)$.

Alternatively, integrate

$$\frac{\partial u}{\partial y} = N$$

wrt y to get

$$u = \int N \, dy + l(x)$$

where $l(x)$ is some fn of x to be determined.

To get l, diff wrt x & use

$$\frac{\partial u}{\partial x} = M \text{ to get } \frac{dl}{dx}.$$

Then integrate $\frac{dl}{dx}$ wrt x to get $l(x)$.

Ex. Solve.

$$\cos(x+y)\,dx + (3y^2 + 2y + \cos(x+y))\,dy = 0$$

Soln

Step 1 Test for exactness.

Here

$$M = \cos(x+y)$$

$$N = 3y^2 + 2y + \cos(x+y)$$

$$\frac{\partial M}{\partial y} = -\sin(x+y)$$

$$\frac{\partial N}{\partial x} = 0 + 0 - \sin(x+y).$$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and the form is exact.

Step 2 Implicit General Sol^{n}.

Have

$$u = \int M\,dx + k(y)$$

$$= \int \cos(x+y)\,dx + k(y)$$

$$= \sin(x+y) + k(y).$$

To find $k(y)$, diff wrt y & use $\frac{\partial u}{\partial y} = N$ to get.

$$\frac{\partial u}{\partial y} = \cos(x+y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x+y)$$

Thus $$\frac{dk}{dy} = 3y^2 + 2y$$

and if we integrate wrt y

$$k = y^3 + y^2 + C.$$

Substituting for k gives.

$$u = \sin(x+y) + y^3 + y^2 + C.$$

Step 3 Checking the answer

If we substitute this into the ODE

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

$$= (\cos(x+y) + 0)\,dx + (\cos(x+y) + 3y^2 + 2y)\,dy$$

$$= 0 \qquad (\text{as } u(x,y) = c)$$

i.e.

$$\cos(x+y)\,dx + (3y^2 + 2y + \cos(x+y))\,dy = 0$$

and so we do have a sol^n as required.

Ex. WARNING - Breakdown in Case of non-Exactness!

$$-y\,dx + x\,dy = 0$$

is not exact as $M = -y$, so $\frac{\partial M}{\partial y} = -1$

but $N = +x$, so $\frac{\partial N}{\partial x} = +1 \neq \frac{\partial M}{\partial y}$.

Note that if we tried to find a sol^{n} anyway, it would not work.

e.g. If we set

$$u = \int M\,dx + k(y)$$

$$= \int -y\,dx + k(y)$$

$$= -xy + k(y).$$

Then diff wrt & use $\frac{\partial u}{\partial y} = N$

$$-x + \frac{dk}{dy} = x$$

Thus

$$\frac{dk}{dy} = 2x$$

which is impossible as the lhs. depends on y only but the rhs is a non-constant fn of x.

Reduction to Exact Form : Integrating Factors

Even though $-y\,dx + x\,dy$ is not exact, if we multiply by $\frac{1}{x^2}$, we do get an exact eqn. (ex- check exactness).

$$\frac{-y\,dx + x\,dy}{x^2} = -\frac{y}{x^2}\,dx + \frac{1}{x}\,dy = d\left(\frac{y}{x}\right) = 0$$

Integration then gives

$$\frac{y}{x} = c$$

or $\quad y = cx \quad$ (lines through $(0,0)$).

IDEA For a given non-exact eqn

$$P(x,y)dx + Q(x,y)dy = 0$$

multiply both sides by a fn F (in general a fn of x & y) to get an eqn.

$$FP\,dx + FQ\,dy = 0$$

which is exact. Such a fn F is called an integrating factor.

N.b. Integrating factors are not in general unique.

e.g. $\frac{1}{y^2}$, $\frac{1}{xy}$, $\frac{1}{x^2+y^2}$

are also all integrating factors for

$$-y\,dx + x\,dy = 0.$$

Finding Integrating Factors

Idea Try to get away with finding a simple integrating factor - it often works!

For $M\,dx + N\,dy$, the exactness condition is

$$M_y = N_x$$ (here the subscripts denote partial derivatives)

For $FP\,dx + FQ\,dy = 0$, the exactness condition is then

$$(FP)_y = (FQ)_x$$

and using the product rule, we get

$$F_y P + F P_y = F_x Q + F Q_x.$$

Try to find an integrating factor which depends on only 1 variable, e.g. $F = F(x)$.

Then $F_y = 0$, $F_x = F' = \frac{dF}{dx}$ & so

$$0 + F P_y = F' Q + F Q_x$$

$$F P_y = F' Q + F Q_x$$

Divide by FQ.

$$\frac{P_y}{Q} = \frac{F'}{F} + \frac{Q_x}{Q}$$

Rearrange:

$$\frac{F'}{F} = \frac{P_y}{Q} - \frac{Q_x}{Q}.$$

$$\frac{1}{F}\frac{dF}{dx} = \frac{1}{Q}(P_y - Q_x)$$

or $\frac{1}{F}\frac{dF}{dx} = R$ where $R = \frac{1}{Q}(P_y - Q_x)$.

Thm 1 (Integrating Factor $F(x)$)

If the ODE $P\,dx + Q\,dy$ is s.t.

$$R = \frac{1}{Q}\left(P_y - Q_x\right) \text{ depends only on } x$$

then there is an integrating factor F which is obtained by integrating

$$\frac{1}{F}\frac{dF}{dx} = R$$

i.e. $$\frac{d}{dx}\left(\ln F\right) = R$$

to get $$\ln F = \int R(x)\,dx$$

$$\Rightarrow \quad F(x) = e^{\int R(x)\,dx}$$

Similarly if F is a fn of y only and we let $F = G(y)$, then

$$\frac{1}{G}\frac{dG}{dy} = S \quad \text{where} \quad S = \frac{1}{P}(Q_x - P_y)$$

and we have.

Thm 2 (Integrating Factor $G(y)$)

If the ODE $P\,dx + Q\,dy$ is s.t.

$$S = \frac{1}{P}(Q_x - P_y) \text{ depends only on } y,$$

then there is an integrating factor $G = G(y)$ which is given by

$$G(y) = e^{\int S(y)\,dy}.$$

n.b. You can get the formulae for G from those for F by

$$x \longleftrightarrow y$$
$$P \longleftrightarrow Q$$

This saves memorizing!

Ex. Using Thm 1 or 2, find an integrating factor and solve

$$(e^{x+y} + ye^{y})\,dx + (xe^{y} - 1)\,dy = 0, \quad y(0) = -1$$

Sol'n. Step 1 Non-exactness.

$$P_y = \frac{\partial}{\partial y}(e^{x+y} + ye^{y}) = e^{x+y} + e^{y} + ye^{y}$$

$$Q_x = \frac{\partial}{\partial x}(xe^{y} - 1) = e^{y}$$

So $P_y \neq Q_x$.

Step 2 Integrating factor

Thm 1 fails because

$$R = \frac{1}{Q}(P_y - Q_x) = \frac{1}{xe^{y}-1}(e^{x+y} + e^{y} + ye^{y} - e^{y})$$

$$= \frac{1}{xe^{y}-1}(e^{x+y} + ye^{y})$$

depends on y as well as x.

Try thm 2.

$$S = \frac{1}{P}(Q_x - P_y) = \frac{1}{e^{x+y} + ye^y}(\cancel{e^y} - e^{x+y} - \cancel{e^y} - ye^y)$$

$$= -1$$

This does not depend on x & gives the integrating factor

$$e^{\int S(y)dy} = e^{\int -dy} = e^{-y}.$$

By Thm 2, the eqn.

$$e^{-y}(e^{x+y} + y e^y)dx + e^{-y}(xe^y - 1) = 0$$

ie

$$(e^x + y)dx + (x - e^{-y}) = 0$$

is then exact. (check this!)

Then by integration

$$u = \int (e^x + y)\,dx = e^x + xy + k(y)$$

Diff w.r.t. y

$$u_y = x + \frac{dk}{dy} = N = x - e^{-y}$$

$$\Rightarrow \quad \frac{dk}{dy} = -e^{-y}$$

$$\Rightarrow \quad k = e^{-y} + c$$

Hence, the general soln is

$$u(x,y) = e^x + xy + e^{-y} = c$$

<u>Step 3</u> Particular soln.

$y(0) = -1$, gives

$u(0,-1) = 1 + 0 + e \approx 3.72$. Hence the

soln is $u(x,y) = e^x + xy + e^{-y} = 3.72$