# § 7.3 Linear Systems

One of the most important uses of matrices.

A <u>linear system</u> of $m$ equations in $n$ unknowns $x_1, x_2, \ldots, x_n$ is a set of equations of the form

$$\begin{array}{ccccccccc}
a_{11} x_1 & + & a_{12} x_2 & + & \cdots & + & a_{1n} x_n & = & b_1 \\
a_{21} x_1 & + & a_{22} x_2 & + & \cdots & + & a_{2n} x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1} x_1 & + & a_{m2} x_2 & + & \cdots & + & a_{mn} x_n & = & b_m
\end{array}$$

The numbers $a_{ij}$ are called the <u>coefficients</u>.

If $b_1 = b_2 = \cdots = b_m = 0$ (i.e. rhs $= 0$), we say the system is <u>homogeneous</u>, otherwise the system is <u>inhomogeneous</u>.

If we let $A$ be the $m \times n$ matrix

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & & \\ a_{m1} & & & a_{mn} \end{bmatrix}$$

and $x$, $b$ be the column vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

then we can write this system in <u>matrix form</u> as

$$\underset{m\times n}{A} \cdot \underset{n\times 1}{x} = \underset{m\times 1}{b}$$

<u>Advantage</u> Matrices are easier to work with than systems of equations.

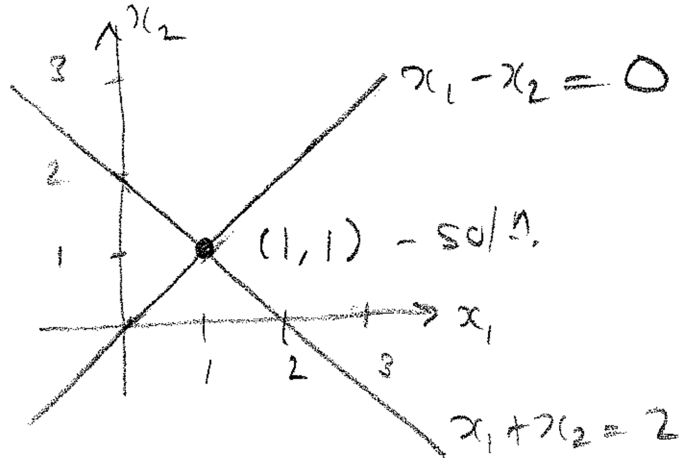
There are 3 basic possibilities

a) Lines Intersect at just one point

— just one (unique) sol$^{n}$.

e.g $x_1 + x_2 = 2$

$x_1 - x_2 = 0$

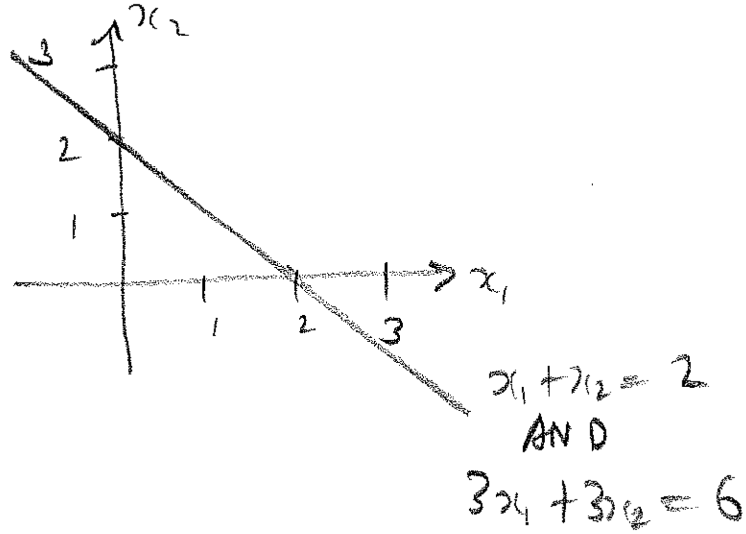



b) Lines lie on top of each other (same line)

- infinitely many sol$^{n}$s.

e.g. $x_1 + x_2 = 2$

$3x_1 + 3x_2 = 6$

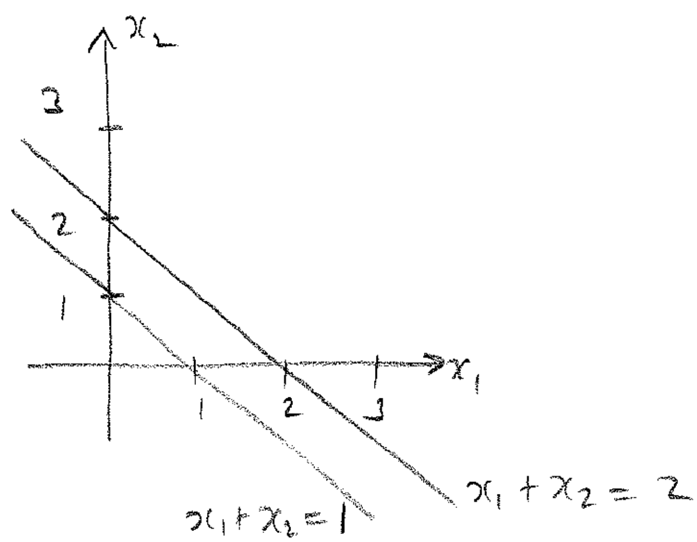


c) Lines do not intersect (parallel and different)

– no $\text{sol}^{\underline{n}}$.

e.g. $x_1 + x_2 = 1$

$x_1 + x_2 = 2$




Similar but more complicated things happen in higher dimensions.

# Gaussian Elimination and Back Substitution

A standard method for solving linear systems which always works.

Suppose we have a system in <u>triangular form</u>

e.g.

$$2x_1 + 5x_2 = 2 \qquad (1)$$

$$13x_2 = -26 \qquad (2)$$

Can solve this by <u>back substitution</u>.

e.g. (2) $\Rightarrow$ $x_2 = -2$

Substitute in (1)

$$2x_1 + 5(-2) = 2$$

$$2x_1 - 10 = 2$$

$$2x_1 = 12$$

$$x_1 = 6.$$

<u>Moral</u> Solving upper triangular systems by back substitution is easy!

Gaussian elimination consists of two steps.

1. Transform the system into an equivalent upper triangular system which has the same sol$^{n}$ (if any).

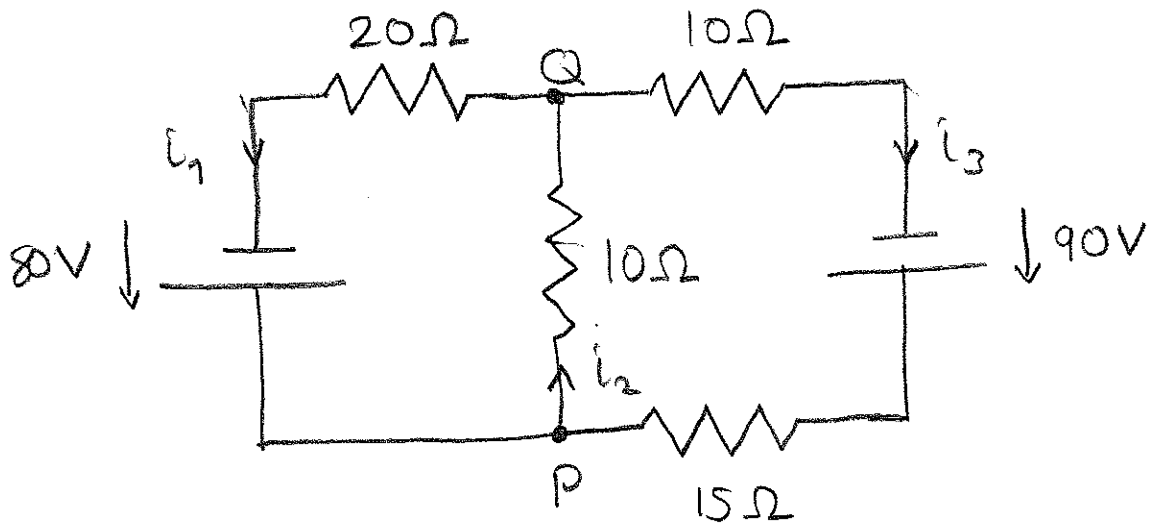
2. Solve the upper triangular system by back substitution which also gives the sol$^{n}$ of the original system.

<u>Ex</u>.

$$\begin{aligned} 2x_1 + 5x_2 &= 2 \\ -4x_1 + 3x_2 &= -30 \end{aligned} \qquad \text{Augmented matrix} \quad \left[\begin{array}{cc|c} 2 & 5 & 2 \\ -4 & 3 & -30 \end{array}\right]$$

Leave the first equation & eliminate $x_1$ from the second equation by adding twice the first equation to it. In the augmented matrix, this corresponds to adding twice the first row to the second.

Ex. Consider the following electrical circuit where we want to find the current in each branch.




The eq$^n$s for the circuit come from Kirchoff's Laws

<u>Kirchoff's Current Law (KCL)</u>

At any pt. in the circuit the sum of the inflowing currents equals the sum of the outflowing currents.

<u>Kirchoff's Voltage Law (KVL)</u>

In any closed loop, the sum of all the voltage drops equals the impressed electromotive force.

KCL at node P: $i_1 - i_2 + i_3 = 0$

KCL at node Q: $-i_1 + i_2 - i_3 = 0$

KVL on right loop: $10 i_2 + 25 i_3 = 90$

KVL on left loop: $20 i_1 + 10 i_2 = 80$.

Using $x$'s instead of $i$'s, we write this more conventionally as

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 \quad &= 80 \end{aligned}$$

***Solution by Gauss Elimination.*** This system could be solved rather quickly by noticing its particular form. But this is not the point. The point is that the Gauss elimination is systematic and will work in general, also for large systems. We apply it to our system and then do back substitution. As indicated let us write the augmented matrix of the system first and then the system itself:

$$\text{Augmented Matrix } \tilde{\mathbf{A}} \qquad\qquad \text{Equations}$$

$$\begin{array}{r} \text{Pivot 1} \longrightarrow \\ \\ \text{Eliminate} \longrightarrow \\ \\ \end{array}\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ \boxed{-1} & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ \boxed{20} & 10 & 0 & 80 \end{array}\right] \qquad \begin{array}{r} \text{Pivot 1} \longrightarrow \\ \\ \text{Eliminate} \longrightarrow \\ \\ \end{array} \begin{array}{rrrl} x_1 - & x_2 + & x_3 = & 0 \\ \boxed{-x_1} + & x_2 - & x_3 = & 0 \\ & 10x_2 + & 25x_3 = & 90 \\ \boxed{20x_1} + & 10x_2 & = & 80. \end{array}$$

***Step 1. Elimination of $x_1$***

Call the first row of **A** the **pivot row** and the first equation the **pivot equation.** Call the coefficient 1 of its $x_1$-term the **pivot** in this step. Use this equation to eliminate $x_1$ (get rid of $x_1$) in the other equations. For this, do:

Add 1 times the pivot equation to the second equation.

Add $-20$ times the pivot equation to the fourth equation.

This corresponds to **row operations** on the augmented matrix as indicated in BLUE behind the ***new matrix*** in (3). So the operations are performed on the ***preceding matrix.*** The result is

$$\text{(3)} \qquad \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right] \quad \begin{array}{l} \\ \text{Row 2 + Row 1} \\ \\ \text{Row 4} - \text{20 Row 1} \end{array} \qquad \begin{array}{rrrl} x_1 - & x_2 + & x_3 = & 0 \\ & & 0 = & 0 \\ & 10x_2 + & 25x_3 = & 90 \\ & 30x_2 - & 20x_3 = & 80. \end{array}$$

***Step 2. Elimination of $x_2$***

The first equation remains as it is. We want the new second equation to serve as the next pivot equation. But since it has no $x_2$-term (in fact, it is $0 = 0$), we must first change the order of the equations and the corresponding rows of the new matrix. We put $0 = 0$ at the end and move the third equation and the fourth equation one place up. This is called **partial pivoting** (as opposed to the rarely used *total pivoting*, in which also the order of the unknowns is changed). It gives

$$\begin{array}{r} \\ \text{Pivot 10} \longrightarrow \\ \text{Eliminate 30} \longrightarrow \\ \\ \end{array}\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & \boxed{30} & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{r} \\ \text{Pivot 10} \longrightarrow \\ \text{Eliminate } 30x_2 \longrightarrow \\ \\ \end{array} \begin{array}{rrrl} x_1 - & x_2 + & x_3 = & 0 \\ & 10x_2 + & 25x_3 = & 90 \\ & \boxed{30x_2} - & 20x_3 = & 80 \\ & & 0 = & 0 \end{array}$$

To eliminate $x_2$, do:

Add $-3$ times the pivot equation to the third equation.

The result is

$$\text{(4)} \qquad \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \\ \\ \text{Row 3} - \text{3 Row 2} \\ \\ \end{array} \qquad \begin{array}{rrrr} x_1 - & x_2 + & x_3 = & 0 \\ & 10x_2 + & 25x_3 = & 90 \\ & & -95x_3 = & -190 \\ & & 0 = & 0 \end{array}$$

***Back Substitution.*** ***Determination of $x_3$, $x_2$, $x_1$ (in this order)***

Working backward from the last to the first equation of this "triangular" system (4), we can now readily find $x_3$, then $x_2$, and then $x_1$:

$$\begin{array}{rrrr} & & -95x_3 = & -190 \\ & 10x_2 + & 25x_3 = & 90 \\ x_1 - & x_2 + & x_3 = & 0 \end{array} \qquad \begin{array}{l} x_3 = i_3 = 2 \text{ [A]} \\ x_2 = \frac{1}{10}(90 - 25x_3) = i_2 = 4 \text{ [A]} \\ x_1 = x_2 - x_3 = i_1 = 2 \text{ [A]} \end{array}$$

where A stands for "amperes." This is the answer to our problem. The solution is unique. ■

# Elementary Row Operations, Row-Equivalent Systems

The previous example used elementary row operations on the augmented matrix. There are 3 kinds

Elementary Row Operations for Matrices

1. Interchange of two rows
2. Addition of a constant multiple of one row to another
3. Multiplication of a row by a non-zero constant $c$.

N.b. These operations are for ROWS, not columns!

In terms of equations, these operations correspond to

Elementary Row Operations for Equations

1. Interchange of two equations
2. Addition of a constant multiple of one equation to another
3. Multiplication of an equation by a non-zero constant $c$.

<u>Important Facts</u> None of these operations changes the solution set of a system of equations and all of them are reversible.

Say two linear systems $S_1, S_2$ (or two matrices $A_1, A_2$) are row equivalent if $S_1$ can be transformed into $S_2$ ($A_1$ into $A_2$) by finitely many elementary row operations on equations (matrices).

We have proved the following justification of Gaussian elimination

<u>Theorem 1</u> Row equivalent linear systems have the same set of sol$^{\underline{n}}$s.

# Row Echelon and Reduced Row Echelon Forms

A matrix $A$ is in <u>row echelon form</u> if:

a) all the rows with all zeroes lie below all the rows which are not all zeroes;

b) the first non-zero entry (pivot) in row $i+1$ lies at least one column to the right of the first non-zero entry (pivot) in row $i$;

c) all the entries below each pivot are zero.

$$\begin{bmatrix} \bullet & \times & \times & \times & \times \\ 0 & \bullet & \times & \times & \times \\ 0 & 0 & 0 & \bullet & \times \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivots

e.g. $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ✓ $\quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix}$ X

A matrix $A$ is in reduced echelon form if it is in row echelon form and in addition:

e) all the pivots are 1;

f) all the entries <u>above</u> each pivot are 0 (as well as all those below).

$$\begin{bmatrix} 1 & 0 & x & 0 & x \\ 0 & 1 & x & 0 & x \\ 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

e.g. $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ✓, $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ X

$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ X

## Facts

Any matrix can be reduced to echelon or reduced row echelon form by elementary row operations

The reduced row echelon form of a matrix is unique, i.e. any matrix is row equivalent to one and only one matrix in reduced echelon form (r.e.f.).

Ex. Gaussian Elimination if no Sol$^{n}$ Exists.

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \qquad \begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 + x_3 &= 0 \\ 6x_1 + 2x_2 + 4x_3 &= 6 \end{aligned}$$

Step 1 (Eliminate $x_1$)
Use the 3 in the top left corner as a pivot.

$R_2 - \frac{2}{3}(\text{Row } 1)$, $R_3 - 2(\text{Row } 1)$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \qquad \begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ -\tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 &= -2 \\ -2x_2 + 2x_3 &= 0 \end{aligned}$$

Step 2 (Eliminate $x_2$).
Use the $-\frac{1}{3}$ in the 2,2 position as a pivot

Row 3 $-$ 6(Row 2)

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] \leftarrow \text{Pivot} \qquad \begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ -\tfrac{1}{3}x_1 + \tfrac{1}{3}x_3 &= -2 \\ 0 &= 12. \end{aligned}$$

The last eq$^{n}$ doesn't make any sense and so there is no sol$^{n}$.

<u>Note:</u> We only needed the echelon form to determine whether there was a sol$^{\Delta}$.

<u>Defn</u> Say a linear system is <u>consistent</u> if it has a sol$^{\Delta}$ (one or infinitely many). Say it is <u>inconsistent</u> if there is no sol$^{\Delta}$.

<u>Fact</u> A linear system is consistent if and only if (iff) there is no pivot in the rightmost column of the augmented matrix for a row equivalent system in row echelon form.

<u>Step 2</u>. Use the 1.1 in the 2,2 position as a pivot (eliminate $x_2$).

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

At this stage we could use back substitution (as the book does).

However, it is better to instead get the reduced echelon form, before finding the sol$^{n}$. (slightly more work but gives a better, more reliable answer).

<u>Step 3</u> Row 2 / 1.1 (sets pivot to 1 in 2nd row).

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Ex. Gaussian Elimination if Infinitely Many Solns Exist

$$3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0$$

$$0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 = 2.7$$

$$1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 = 2.1$$

Augmented matrix is

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

As before, it is enough to work simply by row operations on the augmented matrix without having to also keep track of the equations.

Step 1 Use the 3.0 in the 1,1 position as a pivot. (eliminate $x_1$)

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right]$$

Step 4    Row 1 - 2(Row 2)   (Creates a 0 above the pivot in row 2).

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & 3 & 6 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step 5    (Row 1)/3    (Sets pivot in Row 1 to 1).

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step 6    Convert back to a system of eq^ns.

$$x_1 \qquad\qquad + x_4 = 2$$

$$x_2 + x_3 - 4x_4 = 1$$

or

$$x_1 = -x_4 + 2$$

$$x_2 = -x_3 + 4x_4 - 1$$

Note that $x_1, x_2$ depend on $x_3, x_4$ and we can write our soln in vector form as

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_4 + 2 \\ -x_3 + 4x_4 - 1 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

This is called the <u>parametric vector form</u> of the sol$^{n}$. It is one of the advantages of the r.e.f. as it shows very clearly how the sol$^{n}$ depends on two free parameters $x_3, x_4$ (and so we have infinitely many sol$^{n}$s, one for each possible pair of values of $x_3, x_4$).

Another advantage of using the r.e.f. is the uniqueness mentioned earlier. This means that the above sol$^{n}$ is written in a unique way.

In this example, the variables $x_1$; $x_2$ which correspond to the columns of the augmented matrix where there were pivots are called <u>basic variables</u>.
The variables $x_3, x_4$ which correspond to columns where there is no pivot are called <u>free variables</u>.

<u>Moral</u> Using the r.e.f. and getting the parametric vector form shows how the basic variables are given in terms of the free variables.

<u>Fact</u> A consistent linear system has infinitely many solutions iff there are free variables.

Equivalently, a consistent linear system has only one (unique) sol$^{\text{n}}$ iff there are no free variables.

# Important and Useful Facts

Once we have the row echelon form, we can already tell whether or not a system is consistent.

If the system is consistent, we also know already where the pivots are and whether there are free variables. This can save us the trouble of having to find the r.r.e.f unneccessarily.

# Summary

There are 3 possibilities for the sol$^{\underline{n}}$ of a linear system

1. <u>Inconsistent</u> – Rightmost column of the row echelon form of the augmented matrix contains a pivot

$$\left[\begin{array}{ccccc|c}
\bullet & \times & \times & \times & \times & \times \\
0 & \bullet & \times & \times & \times & \times \\
0 & 0 & 0 & \bullet & \times & \times \\
0 & 0 & 0 & 0 & 0 & \bullet \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\begin{array}{l} \\ \\ \\ \longleftarrow \text{pivot.} \\ \\ \end{array}$$

2. <u>Consistent, Unique Soln</u>

$$\left[\begin{array}{cccc|c}
\bullet & \times & \times & \times & \times \\
0 & \bullet & \times & \times & \times \\
0 & 0 & \bullet & \times & \times \\
0 & 0 & 0 & \bullet & \times \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]$$

Here $m \geq n$ (at least as many eq$^{\underline{ns}}$ as unknowns)

Only non-pivot column is the rightmost one.

Consistent, no free variables.

# 3. Consistent, Infinitely Many Sol^{ns}

$$
\begin{array}{cccccc}
\left[\begin{array}{ccccc|c}
0 & \bullet & \times & \times & \times & \times \\
0 & 0 & \bullet & \times & \times & \times \\
0 & 0 & 0 & 0 & \bullet & \times \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \\
\begin{array}{ccccc}
x_1 & x_2 & x_3 & x_4 & x_5
\end{array}
\end{array}
$$

Here there are pivots in cols 2, 3, 5.
No pivot in rightmost col., so system is consistent.

$x_2, x_3, x_5$ are <u>basic</u>

$x_1, x_5$ are <u>free</u>.