§ 1.3 Separable ODEs Modelling

Many common ODEs can be reduced to the form

$$g(y)\, y' = f(x). \qquad (1)$$

by algebraic manipulation.

We can then integrate both sides WRT x to get

$$\int g(y) \cdot y' \, dx = \int f(x)\, dx + c$$

On the lhs, if we use the subst. $y = y(x)$, so $dy = \frac{dy}{dx} \cdot dx = y' \, dx$, we get.

$$\int g(y)\, dy = \int f(x)\, dx + c.$$

Doing the integrals allows us to (implicitly) get our sol^n y as a f^n of x. This method is called <u>separating the variables</u> and an eq^n such as (1) is called a <u>separable eqn</u>.

Ex.

$$y' = 1 + y^2$$

This is separable as we can write

$$\frac{dy}{1+y^2} = dx$$

$$\int \frac{1}{1+y^2}\, dy = \int dx$$

$g(y) = \frac{1}{1+y^2}$ $\qquad$ $f(x) = 1$.

$$\arctan y = x + c$$

or

$$y = \tan(x+c).$$

N.b. It is important to put in the const of integration c <u>immediately after integration</u>. Putting it in later would give us instead $y = \tan x + c$ which is not a soln (if $c \neq 0$).

Modelling

Ex. A radioactive material decays at a rate proportional to the number of radioactive atoms present.

A sample of organic material is found to have 52·5% of the radioactive C^{14} of living tissue (in which the amount of C^{14} stays constant).

Given that the half-life of C^{14} is 5715 years, find the age of the sample.

Sol^n. Let $y(t)$ be the amount of C^{14} present as a fn of time t (in years).

Have $y' = ky$ for some const k (decay const).

so
$$\frac{dy}{y} = k\,dt$$

$$\int \frac{dy}{y} = \int k\,dt$$

$$\ln|y| = kt + c$$

and since $y > 0$

$$\ln y = kt + c$$

or $y = e^{kt+c} = e^{c} \cdot e^{kt}$

If we start with an amount y_0 at $t=0$, then

$$y_0 = e^{c} e^{k \cdot 0} = e^{c}$$

So $y = y_0 e^{kt}$

Still need to find k. Use the half-life.
If we start with y_0, then after 5715 years we have $\frac{y_0}{2}$ left. Thus

$$\frac{y_0}{2} = y_0 e^{k \cdot 5715}$$

$$\frac{1}{2} = e^{5715k}$$

Take ln of both sides.

$$\ln\left(\frac{1}{2}\right) = 5715k$$

So $k = \dfrac{\ln\left(\frac{1}{2}\right)}{5715} \approx -0.0001213$.

Finally, to find the age of the sample.

$$.525 y_0 = y_0 e^{-.0001213 \cdot t}$$

$$.525 = e^{-.0001213\,t}$$

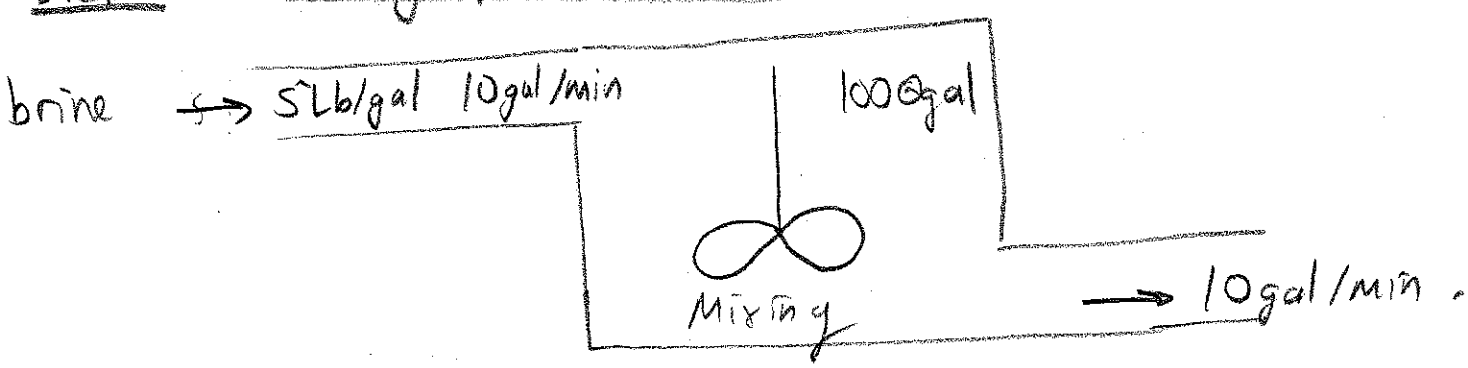
$$\ln(.525) = -.0001213\,t$$

$$t = \frac{\ln(.525)}{.0001213} \approx 5312 \text{ years.}$$

Ex Mixing.

A tank contains 1000 gal of water in which initially 100kg of salt is dissolved. Brine in which 5 lb of salt is dissolved in each gallon, flows into the tank at 10 gal/min while the tank is also being emptied at a rate of 10 gal/min. Find the amount of salt in the tank at any given time t.

Step 1 Setting up the Model



Let $y = y(t)$ be the amount of salt in the tank t.

Then

$y' =$ Salt inflow rate $-$ Salt outflow rate

Step 2 Solution of the Model.

This ODE is separable. Get

$$\frac{dy}{y-5000} = -.01dt$$

$$\int \frac{dy}{y-5000} = \int -.01dt$$

$$\ln|y-5000| = -.01t + k \quad , \quad k \text{ const.}$$

$$y - 5000 = e^{-.01t+k}$$

$$= e^{k}e^{-.01t}$$

$$= ce^{-.01t} \quad , \quad c = e^{k}$$

$$y = 5000 + ce^{-.01t}$$

Use initial condition to get c

$y(0) = 100$, so

In 1 minute

10 gal of brine enters with 5 lb of salt dissolved in each gallon.

$\Rightarrow$ 50 lbs of salt enter.

10 gal of sol^n leaves.
If there are y lbs in the tank, the amount of salt lost is

$$\frac{10}{1000} = .01$$

of the total, so $.01y$ lbs of salt leaves.

Thus, we have the IVP

$$y' = 50 - .01y = -.01(y - 5000)$$

$$y(0) = 100.$$

$$100 = 5000 + c\,e^{0}$$

$$= 5000 + c$$

$$-4900 = c$$

So

$$y(t) = 5000 - 4900\,e^{-.01t} \text{ lbs.}$$

Reduction to Separable Form

Suppose we have an ODE of the form

$$y' = f\left(\frac{y}{x}\right).$$ $\left(\text{eg. } \frac{x^4}{y^4},\ \sin\left(\frac{x}{y}\right),\ \frac{y}{x}\right)$

If we set $\frac{y}{x} = u$, then

$$y = ux$$

so by the product rule

$$\begin{aligned} y' &= u'x + ux' \\ &= u'x + u \end{aligned}$$

and if we substitute in the ODE

$$u'x + u = f(u)$$

or

$$u'x = f(u) - u$$

and we can separate this to get

$$\frac{du}{f(u)-u} = \frac{dx}{x}.$$

<u>MORAL</u> The change of variables $y \to u$ gives us a separable equation which is easier to solve. We then convert back to y (ie $y = ux$) to get our desired soln y.

This is a very common approach to solving ODEs.

Ex. $$2xyy' = y^2 - x^2$$

Divide by $2xy$

$$y' = \frac{y^2}{2xy} - \frac{x^2}{2xy}$$

$$= \frac{y}{2x} - \frac{x}{2y} \qquad = \frac{1}{\left(\frac{y}{x}\right)} = \frac{1}{u}$$

Now let $u = \frac{y}{x}$ and $y' = u'x + u$ as before to get

$$u'x + u = \frac{u}{2} - \frac{1}{2u}$$

$$u'x = -\frac{u}{2} - \frac{1}{2u}$$

$$= \frac{-u^2 - 1}{2u}$$

Separate $$\frac{2u\,du}{1+u^2} = -\frac{dx}{x}.$$

$$\int \frac{2u\,du}{1+u^2} = -\int \frac{dx}{x}$$

$v = 1+u^2, \quad dv = 2u\,du$

$$\int \frac{dv}{v} = -\int \frac{dx}{x}$$

$$\ln|v| = -\ln|x| + k$$

$$\ln|1+u^2| = \ln\left(\frac{1}{|x|}\right) + k$$

$$\ln(1+u^2) = \ln\left(\frac{1}{|x|}\right) + k$$

(n.b. $1+u^2 > 0$ so $|1+u^2| = 1+u^2$)

Take exp of both sides

$$1+u^2 = e^{\ln\left(\frac{1}{|x|}\right)+k}$$

$$= e^k \cdot \frac{1}{|x|}$$

$$= \frac{c}{x} \qquad c = \pm e^k.$$

Convert back to y.

$$1 + \left(\frac{y}{x}\right)^2 = \frac{c}{x}$$

Multiply both sides by x^2

$$x^2 + y^2 = cx$$

$$x^2 - cx + y^2 = 0$$

Add $\frac{c^2}{4}$ to both sides to complete the square

$$x^2 - cx + \frac{c^2}{4} + y^2 = \frac{c^2}{4}$$

$$\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$$

eqn of circle of radius $\frac{c}{2}$ and centre $\left(\frac{c}{2}, 0\right)$