Chapter 1

First Order ODE's

§ 1.1 Basic Concepts, Modelling

Many scientific problems are modelled by differential equations

Examples Electric circuits, mechanical systems, population dynamics, response to a drug.

An ordinary differential equation (ODE) is an equation involving one or more derivatives of a fn $y = y(x)$ of a single variable (n.b. y itself may appear in the eqn as the 0^{th} order derivative, but there must be derivatives of order ≥ 1 appearing).

Examples.

1) $y' = \cos x$

2) $y'' + 9y = 0$

3) $x^2 y''' y' + 2e^x y'' = (x^2+2) y^2$

The order of an ODE is simply the order of the highest order derivative of y appearing in the eqn.

e.g. 1) is a first order
2) is second order
3) is third order

For now, we will concern ourselves with first order ODE's which can be expressed in the form

$$F(x, y, y') = 0.$$

Often (but by no means always), we can get y' on its own and write the ODE as

$$y' = f(x, y).$$

<u>N.b.</u> There are also <u>partial differential eqns</u> (PDEs) where y is a fn of more than one variable and we have partial derivatives of y appearing.

e.g.
$$\frac{\partial^2 y}{\partial x_1^2} + \frac{\partial^2 y}{\partial x_2^2} = 0 \quad \text{(Laplace's Eqn).}$$

<u>Notation</u> - Intervals

$(a, b) = \{x \in \mathbb{R} : a < x < b\}$ open interval

$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ closed interval

$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ half open, half closed intervals.

$(a, \infty) = \{x \in \mathbb{R} : a < x\}$
$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$ infinite intervals.

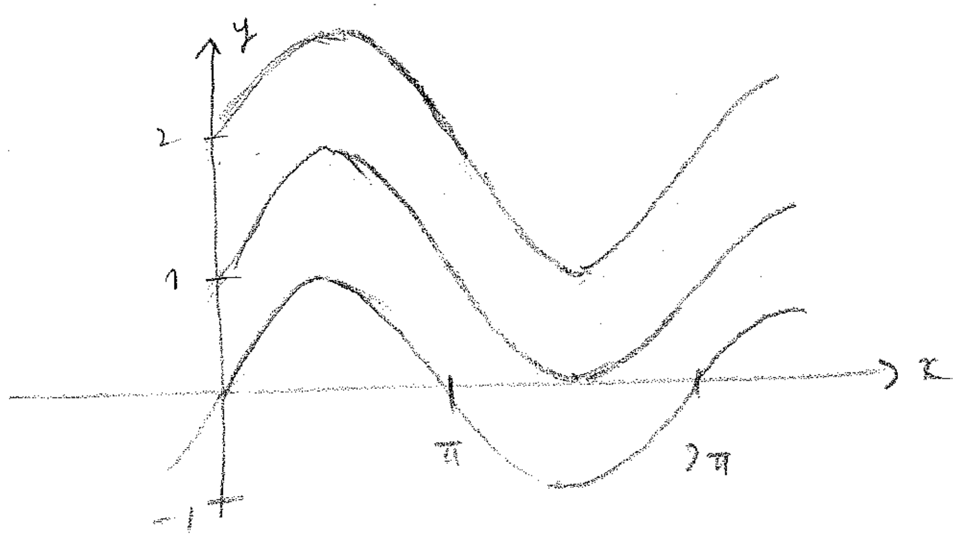
Indeed, if we just integrate both sides of

$$y' = \cos x$$

$$\int y' \, dx = \int \cos x \, dx$$

$$y = \sin x + c$$

where c is a constant of integration.



Solutions of ODEs

A function $y = h(x)$ is a solution of the ODE

$$F(x, y, y') = 0$$

on an open interval (a, b) (or $a < x < b$) if

$$F(x, h(x), h'(x)) = 0, \quad a < x < b.$$

for each x with $a < x < b$.

e.g. $y = h(x) = \sin x$ is a soln of

$$y' = \cos x$$

as $\frac{d}{dx}(\sin x) = \cos x$.

$\sin x + 1$, $\sin x - \frac{\pi^2}{6}$ are also solns as is $\sin x + c$ for any constant c.

Ex. If $y = ce^{3t}$ (c any const)., then

$$y' = \frac{dy}{dt} = 3ce^{3t} = 3y$$

So y is a sol$^{\underline{n}}$ of

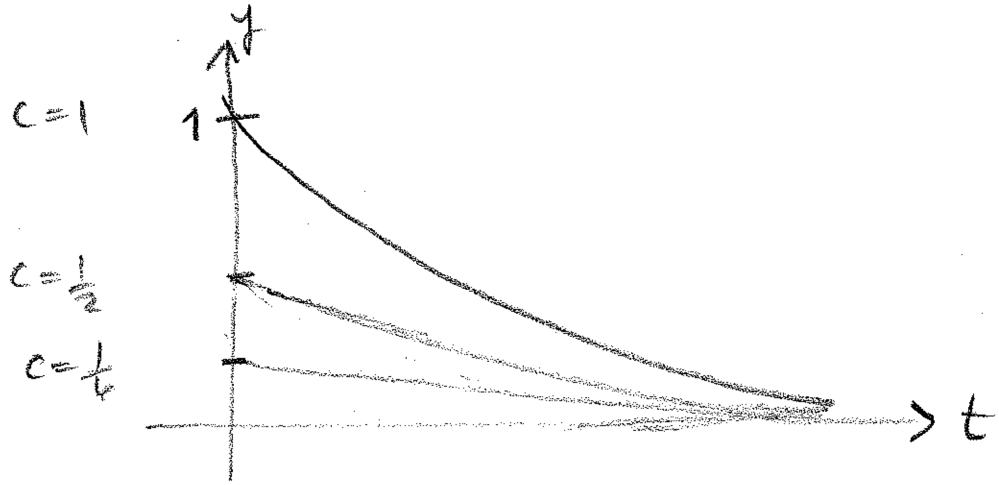
$$y' = 3y$$

This eq$^{\underline{n}}$ models <u>exponential growth</u>.

Similarly $y = ce^{-2t}$ is a sol$^{\underline{n}}$ of

$$y' = -2y.$$

This eq$^{\underline{n}}$ models <u>exponential decay</u>.



In each of the last examples the solⁿ of the ODE contained an arbitrary const.

Such a solⁿ with an arbitrary const is called the <u>general solution</u> of the ODE.

It is actually a whole family of functions which differ from each other by an arb. const.

Taking a particular value of the const. gives us a <u>particular solution</u> of the ODE.

Initial Value Problems

If we require that our soln $y(x)$ of the ODE $y' = f(x,y)$ has a particular value y_0 at the point $x = x_0$ we (usually) get a unique soln.

A problem of this type

$$y' = f(x,y), \qquad y(x_0) = y_0$$

is called an <u>initial value problem</u> (IVP) (y_0 is the initial value of the soln at $x = x_0$).

<u>Ex.</u> For the IVP

$$y' = 3y, \qquad y(0) = 4$$

$y = 4e^{3x}$ is a soln as $y(0) = 4e^0 = 4$.

It is in fact the only soln which satisfies this initial condition.

This can be seen quite readily by letting $z = z(x)$ be any other soln.

Then $z' = 3z$, $z(0) = 4$

and if we let $w = \frac{z}{y}$ (note $y = 4e^{3x}$ $\neq 0 \ \forall x$)

then

$$w' = \frac{yz' - y'z}{y^2} = \frac{y \cdot 3z - 3y \cdot z}{y^2} = 0$$

Thus w is a constant and since

$w(0) = \frac{z(0)}{y(0)} = \frac{4}{4} = 1$, we have

$$w(x) = \frac{z(x)}{y(x)} = 1$$

and so $z(x) = y(x)$ for all $x \in \mathbb{R}$.

Modelling

Using maths to describe, quantify, and predict real-world phenomena.

3 Main Steps

Step 1 Turn the physical into a mathematical formulation.

Step 2 Solve the mathematical problem

Step 3 Give a physical interpretation of the mathematical answer.

Ex. Given 0.5g of a radioactive substance, find the amount present at any given time.

Experiments show the substance decays at a rate proportional to the amount present.

Step 1. Let $y(t)$ be the amount present (in grams) at time t.

Then, $y' = \frac{dy}{dt}$ is proportional to y and so

$$\frac{dy}{dt} = ky,$$

for some constant k (which is usually known from experiments).

Note that $y > 0$, but $\frac{dy}{dt} < 0$ as the amount present decreases over time.

Thus k is negative.

Since we start ($t=0$) with 0.5g,
we also have the initial condition $y(0)=0.5$.

In summary, we have the IVP

$$y' = ky, \quad y(0) = 0.5$$

Step 2 Mathematical Solution

As in the previous example, we can find that $y(t) = ce^{kt}$ for some const c.

Since $y(0) = 0.5$, this implies

$$0.5 = y(0) = ce^{k \cdot 0}$$

$$0.5 = c \cdot 1$$

$$0.5 = c,$$

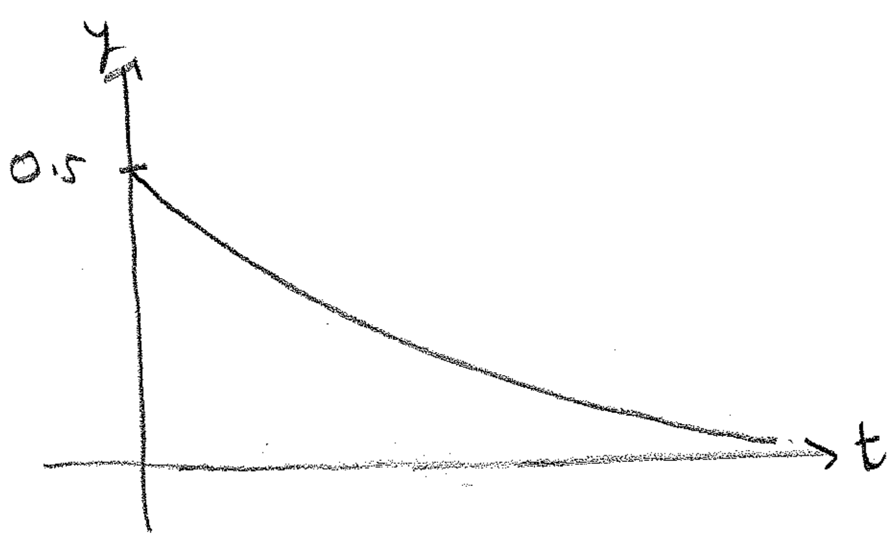
So $y(t) = 0.5e^{kt}$

n.b. One can check by differentiation & substitution that this fn does indeed satisfy the IVP. (One should always do this!)

Step 3 Interpretation.

$y(t)$ starts at value 0.5 and decreases over time with $y(t) \to 0$ as $t \to \infty$.

This means that the substance decays over time with the amount left becoming arbitrarily small as time passes.



Ex. Geometric Application

Find the curve in the xy plane which passes through $(1, 1)$ and has slope $-\frac{y}{x}$.

Soln. We have the IVP

$$y' = -\frac{y}{x} \quad , \quad y(1) = 1.$$

We will see later that the soln of this ODE is a curve of the form

$$y = cx$$

and since $y(1) = 1$, we have

$$1 = c \cdot 1 \quad , \quad \text{so}$$

$$y = x$$

which is one part of a hyperbola.