Section 5.1 Eigenvectors & Eigenvalues

The basic concepts presented here - *eigenvectors* and *eigenvalues* - are useful throughout pure and applied mathematics. Eigenvalues are also used to study difference equations and *continuous* dynamical systems. They provide critical information in engineering design, and they arise naturally in such fields as physics and chemistry.

EXAMPLE: Let $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Examine the images of **u** and **v** under multiplication by A.

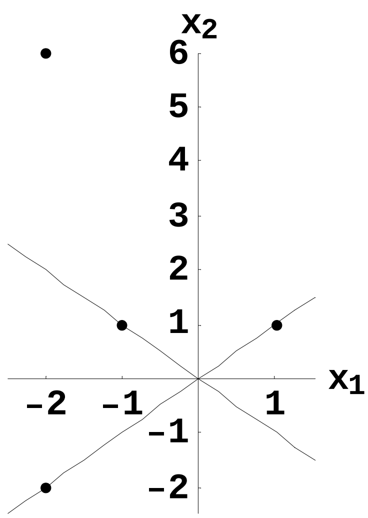
Solution

$$A\mathbf{u} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2\mathbf{u}$$

$$A\mathbf{v} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \neq \lambda\mathbf{v}$$

u is called an *eigenvector* of A.

v is not an eigenvector of A since $A\mathbf{v}$ is not a multiple of **v**.



$A\mathbf{u} = -2\mathbf{u}$, but $A\mathbf{v} \neq \lambda\mathbf{v}$

DEFINITION

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector **x** such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution **x** of $A\mathbf{x} = \lambda\mathbf{x}$; such an **x** is called an *eigenvector corresponding to* λ.

EXAMPLE: Show that 4 is an eigenvalue of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.

Solution: Scalar 4 is an eigenvalue of A if and only if $A\mathbf{x} = 4\mathbf{x}$ has a nontrivial solution.

$$A\mathbf{x} - 4\mathbf{x} = \mathbf{0}$$

$$A\mathbf{x} - 4(\underline{\qquad})\mathbf{x} = \mathbf{0}$$

$$(A - 4I)\mathbf{x} = \mathbf{0}.$$

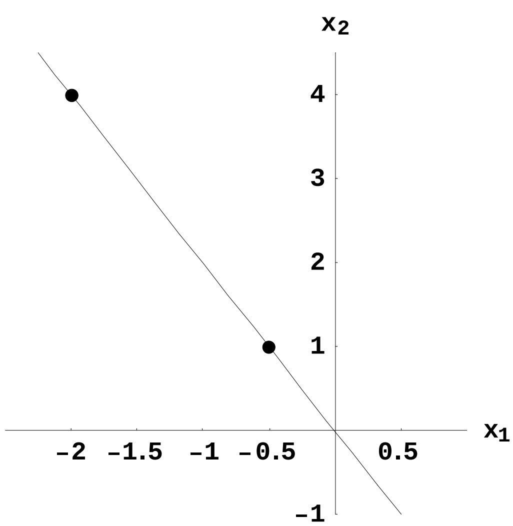
To solve $(A - 4I)\mathbf{x} = \mathbf{0}$, we need to find $A - 4I$ first:

$$A - 4I = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -4 & -2 \end{bmatrix}$$

Now solve $(A - 4I)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -4 & -2 & 0 \\ -4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} \frac{-1}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{-1}{2} \\ 1 \end{bmatrix}.$$

Each vector of the form $x_2 \begin{bmatrix} \frac{-1}{2} \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 4$.



Eigenspace for $\lambda = 4$

Warning: The method just used to find eigen*vectors* ***cannot*** be used to find eigen*values*.

The set of all solutions to $(A - \lambda I)\mathbf{x} = \mathbf{0}$ is called the **eigenspace** of A corresponding to λ.

EXAMPLE: Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$. An eigenvalue of A is $\lambda = 2$. Find a basis for the corresponding eigenspace.

Solution:

$$A - 2I = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} \_\_\_ & 0 & 0 \\ 0 & \_\_\_ & 0 \\ 0 & 0 & \_\_\_ \end{bmatrix}$$

$$= \begin{bmatrix} 2 - \_\_\_ & 0 & 0 \\ -1 & 3 - \_\_\_ & 1 \\ -1 & 1 & 3 - \_\_\_ \end{bmatrix} = \begin{bmatrix} \_\_\_ & 0 & 0 \\ -1 & \_\_\_ & 1 \\ -1 & 1 & \_\_\_ \end{bmatrix}$$

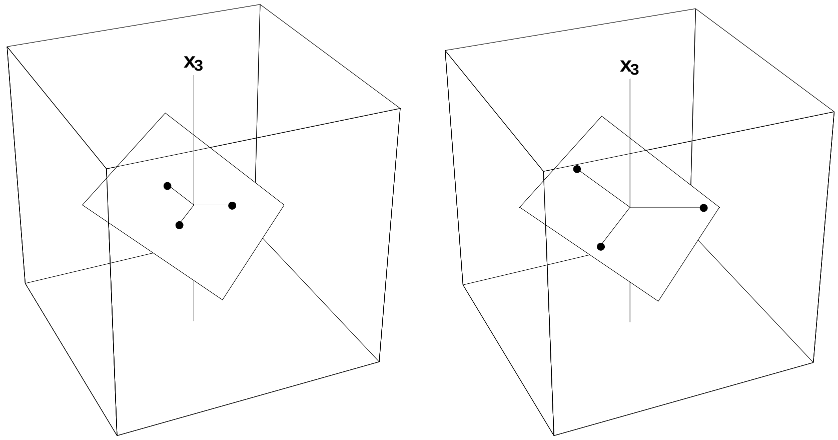
Augmented matrix for $(A - 2I)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = \_\_\_\_ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

So a basis for the eigenspace corresponding to $\lambda = 2$ is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$



Effects of Multiplying Vectors in Eigenspaces for $\lambda = 2$ by A

EXAMPLE: Suppose λ is eigenvalue of A. Determine an eigenvalue of A^2 and A^3. In general, what is an eigenvalue of A^n?

Solution: Since λ is eigenvalue of A, there is a nonzero vector **x** such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Then

$$\_\_\_A\mathbf{x} = \_\_\_\lambda\mathbf{x}$$

$$A^2\mathbf{x} = \lambda A\mathbf{x}$$

$$A^2\mathbf{x} = \lambda\_\_\_\mathbf{x}$$

$$A^2\mathbf{x} = \lambda^2\mathbf{x}$$

Therefore λ^2 is an eigenvalue of A^2.

Show that λ^3 is an eigenvalue of A^3:

$$\_\_\_A^2\mathbf{x} = \_\_\_\lambda^2\mathbf{x}$$

$$A^3\mathbf{x} = \lambda^2 A\mathbf{x}$$

$$A^3\mathbf{x} = \lambda^3\mathbf{x}$$

Therefore λ^3 is an eigenvalue of A^3.

In general, ______ is an eigenvalue of A^n.

THEOREM 1 The eigenvalues of a triangular matrix are the entries on its main diagonal.

Proof for the 3×3 *Upper Triangular Case*: Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}.$$

and then

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}.$$

By definition, λ is an eigenvalue of A if and only if $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution. This occurs if and only if $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a free variable.

When does this occur?

THEOREM 2 If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a linearly independent set.

See the proof on page 307.