4.4 Coordinate Systems

In general, people are more comfortable working with the vector space $\mathbf{R}^n$ and its subspaces than with other types of vectors spaces and subspaces. The goal here is to *impose* coordinate systems on vector spaces, even if they are not in $\mathbf{R}^n$.

THEOREM 7 The Unique Representation Theorem

Let $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

DEFINITION

Suppose $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis β** (or the **β – coordinates of x**) are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

In this case, the vector in $\mathbf{R}^n$

$$[\mathbf{x}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

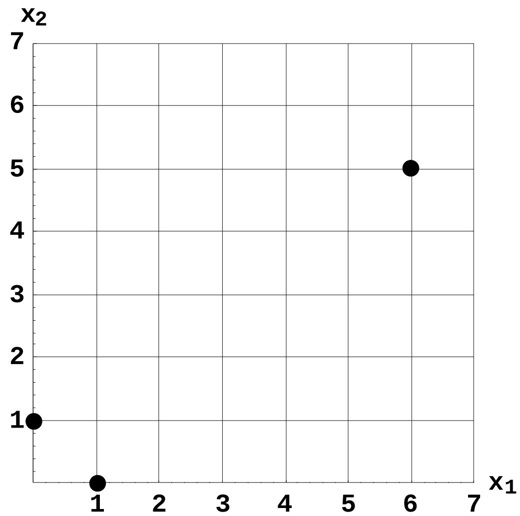
is called the **coordinate vector of x** (**relative to β**), or the **β – coordinate vector of x**.

EXAMPLE: Let $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$ where $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and let $E = \{\mathbf{e}_1, \mathbf{e}_2\}$ where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

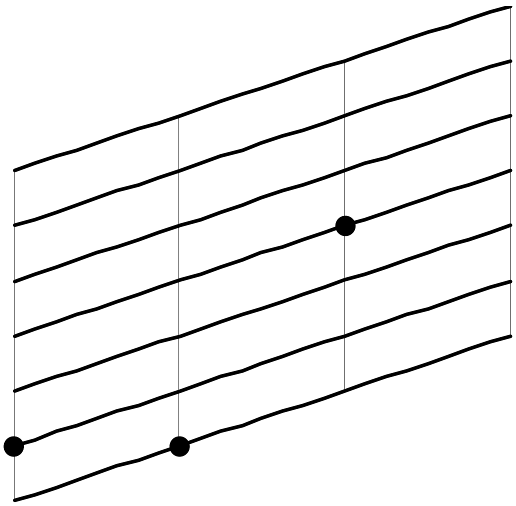
Solution:

If $[\mathbf{x}]_\beta = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then $\mathbf{x} =$ ____ $\begin{bmatrix} 3 \\ 1 \end{bmatrix} +$ ____ $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$.

If $[\mathbf{x}]_E = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$, then $\mathbf{x} =$ ____ $\begin{bmatrix} 1 \\ 0 \end{bmatrix} +$ ____ $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$.



Standard graph paper


β – graph paper

From the last example,

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

For a basis $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, let

$$P_\beta = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix} \text{ and } [\mathbf{x}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Then

$$\mathbf{x} = P_\beta [\mathbf{x}]_\beta.$$

We call P_β the **change-of-coordinates matrix** from β to the standard basis in $\mathbf{R}^n$. Then

$$[\mathbf{x}]_\beta = P_\beta^{-1}\mathbf{x}$$

and therefore P_β^{-1} is a **change-of-coordinates matrix** from the standard basis in $\mathbf{R}^n$ to the basis β.

EXAMPLE: Let $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathbf{x} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$. Find the change-of-coordinates matrix P_β from β to the standard basis in $\mathbf{R}^2$ and change-of-coordinates matrix P_β^{-1} from the standard basis in $\mathbf{R}^2$ to β.

Solution $\quad P_\beta = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$ and so $P_\beta^{-1} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}$

(b) If $\mathbf{x} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$, then use P_β^{-1} to find $[\mathbf{x}]_\beta = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$.

Solution: $\quad [\mathbf{x}]_\beta = P_\beta^{-1}\mathbf{x} = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$

Coordinate mappings allow us to introduce coordinate systems for unfamiliar vector spaces.

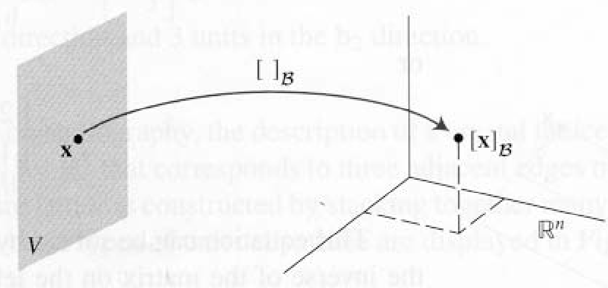


FIGURE 5 The coordinate mapping from V onto $\mathbb{R}^n$.

Standard basis for $\mathbf{P}_2$: $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} = \{1, t, t^2\}$

Polynomials in $\mathbf{P}_2$ behave like vectors in $\mathbf{R}^3$. Since $a + bt + ct^2 =$ ____$\mathbf{p}_1 +$ ____$\mathbf{p}_2 +$ ____$\mathbf{p}_3$,

$$[a + bt + ct^2]_\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

We say that the vector space $\mathbf{R}^3$ is *isomorphic* to $\mathbf{P}_2$.

EXAMPLE: Parallel Worlds of $\mathbf{R}^3$ and $\mathbf{P}_2$.

Vector Space $\mathbf{R}^3$	**Vector Space $\mathbf{P}_2$**
Vector Form: $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$	Vector Form: $a + bt + bt^2$
Vector Addition Example	*Vector Addition Example*
$\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$	$(-1 + 2t - 3t^2) + (2 + 3t + 5t^2)$ $= 1 + 5t + 2t^2$

Informally, we say that vector space V is **isomorphic** to W if *every vector space calculation in V is accurately reproduced in W*, and vice versa.

Assume β is a basis set for vector space V. Exercise 25 (page 254) shows that a set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ in V is linearly independent if and only if $\left\{[\mathbf{u}_1]_\beta, [\mathbf{u}_2]_\beta, \ldots, [\mathbf{u}_p]_\beta\right\}$ is linearly independent in $\mathbf{R}^n$.

EXAMPLE: Use coordinate vectors to determine if $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a linearly independent set, where $\mathbf{p}_1 = 1 - t$, $\mathbf{p}_2 = 2 - t + t^2$, and $\mathbf{p}_3 = 2t + 3t^2$.

Solution: The standard basis set for $\mathbf{P}_2$ is $\beta = \{1, t, t^2\}$. So

$$[\mathbf{p}_1]_\beta = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}, \; [\mathbf{p}_2]_\beta = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}, \; [\mathbf{p}_3]_\beta = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

By the IMT, $\left\{[\mathbf{p}_1]_\beta, [\mathbf{p}_2]_\beta, [\mathbf{p}_3]_\beta\right\}$ is linearly ______________________ and therefore

$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly ______________________.

Coordinate vectors also allow us to associate vector spaces with subspaces of other vectors spaces.

EXAMPLE Let $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$ where $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ and let $H = \text{span}\{\mathbf{b}_1, \mathbf{b}_2\}$.

Find $[\mathbf{x}]_\beta$, if $\mathbf{x} = \begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix}$.

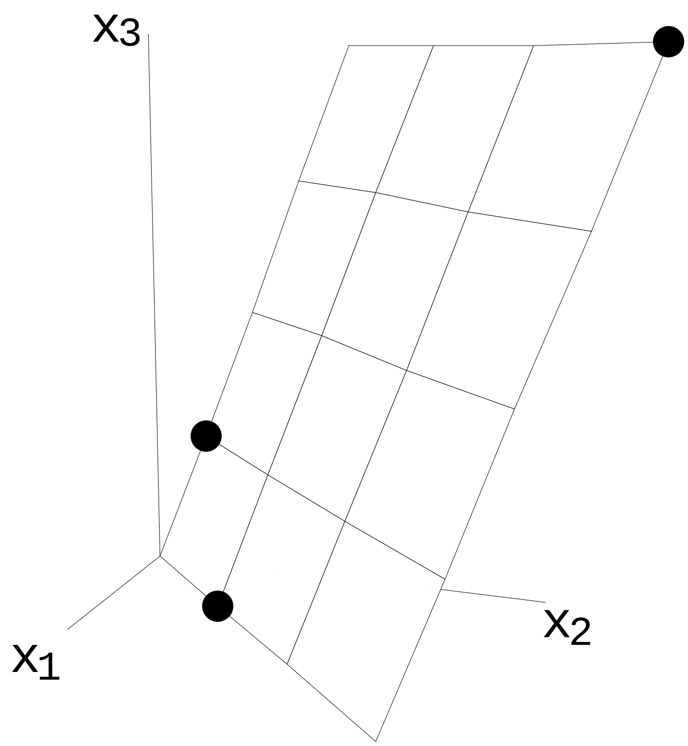
Solution: (a) Find c_1 and c_2 such that

$$c_1 \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix}$$

Corresponding augmented matrix:

$$\begin{bmatrix} 3 & 0 & 9 \\ 3 & 1 & 13 \\ 1 & 3 & 15 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $c_1 =$ ____ and $c_2 =$ _____ and so $[\mathbf{x}]_\beta = \begin{bmatrix} \quad \end{bmatrix}$.



$$\begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix} \text{ in } \mathbf{R}^3 \text{ is associated with the vector } \begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ in } \mathbf{R}^2$$

H is isomorphic to $\mathbf{R}^2$