

4.2 Null Spaces, Column Spaces, & Linear Transformations

The **null space** of an $m \times n$ matrix A , written as $\text{Nul } A$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\text{Nul } A = \{ \mathbf{x} : \mathbf{x} \text{ is in } \mathbf{R}^n \text{ and } A\mathbf{x} = \mathbf{0} \} \quad (\text{set notation})$$

THEOREM 2

The null space of an $m \times n$ matrix A is a subspace of \mathbf{R}^n . Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of \mathbf{R}^n .

Proof: $\text{Nul } A$ is a subset of \mathbf{R}^n since A has n columns. Must verify properties a, b and c of the definition of a subspace.

Property (a) Show that $\mathbf{0}$ is in $\text{Nul } A$. Since _____, $\mathbf{0}$ is in _____.

Property (b) If \mathbf{u} and \mathbf{v} are in $\text{Nul } A$, show that $\mathbf{u} + \mathbf{v}$ is in $\text{Nul } A$. Since \mathbf{u} and \mathbf{v} are in $\text{Nul } A$,
_____ and _____.

Therefore

$$A(\mathbf{u} + \mathbf{v}) = \text{_____} + \text{_____} = \text{_____} + \text{_____} = \text{_____}.$$

Property (c) If \mathbf{u} is in $\text{Nul } A$ and c is a scalar, show that $c\mathbf{u}$ is in $\text{Nul } A$:

$$A(c\mathbf{u}) = \text{___}A(\mathbf{u}) = c\mathbf{0} = \mathbf{0}.$$

Since properties a, b and c hold, A is a subspace of \mathbf{R}^n .

Solving $A\mathbf{x} = \mathbf{0}$ yields an **explicit description** of $\text{Nul } A$.

EXAMPLE: Find an explicit description of $\text{Nul } A$ where $A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$.

Solution: Row reduce augmented matrix corresponding to $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}$$

Then

$$\text{Nul } A = \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$$

Observations:

1. Spanning set of $\text{Nul } A$, found using the method in the last example, is automatically linearly independent:

$$c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = \underline{\hspace{2cm}} \quad c_2 = \underline{\hspace{2cm}} \quad c_3 = \underline{\hspace{2cm}}$$

2. If $\text{Nul } A \neq \{\mathbf{0}\}$, the the number of vectors in the spanning set for $\text{Nul } A$ equals the number of free variables in $A\mathbf{x} = \mathbf{0}$.

The **column space** of an $m \times n$ matrix A ($\text{Col } A$) is the set of all linear combinations of the columns of A .

If $A = [\mathbf{a}_1 \dots \mathbf{a}_n]$, then

$$\text{Col } A = \text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$$

THEOREM 3

The column space of an $m \times n$ matrix A is a subspace of \mathbf{R}^m .

Why? (Theorem 1, page 221)

Recall that if $A\mathbf{x} = \mathbf{b}$, then \mathbf{b} is a linear combination of the columns of A . Therefore

$$\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbf{R}^n\}$$

EXAMPLE: Find a matrix A such that $W = \text{Col } A$ where $W = \left\{ \begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} : x, y \text{ in } \mathbf{R} \right\}$.

Solution:

$$\begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} & \\ & \\ & \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{Therefore } A = \begin{bmatrix} & \\ & \\ & \end{bmatrix}.$$

By Theorem 4 (Chapter 1),

The column space of an $m \times n$ matrix A is all of \mathbf{R}^m if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each \mathbf{b} in \mathbf{R}^m .

The Contrast Between Nul A and Col A

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 0 & 0 & 1 \end{bmatrix}$.

- (a) The column space of A is a subspace of \mathbf{R}^k where $k = \underline{\hspace{2cm}}$.
- (b) The null space of A is a subspace of \mathbf{R}^k where $k = \underline{\hspace{2cm}}$.
- (c) Find a nonzero vector in Col A . (There are infinitely many possibilities.)

$$\underline{\hspace{1cm}} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \underline{\hspace{1cm}} \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix} + \underline{\hspace{1cm}} \begin{bmatrix} 3 \\ 7 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

- (d) Find a nonzero vector in Nul A . Solve $A\mathbf{x} = \mathbf{0}$ and pick one solution.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 7 & 0 \\ 3 & 6 & 10 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -2x_2$$

x_2 is free

$$x_3 = 0$$

Let $x_2 = \underline{\hspace{1cm}}$ and then

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Contrast Between Nul A and Col A where A is $m \times n$ (see page 232)

Review

A **subspace** of a vector space V is a subset H of V that has three properties:

- The zero vector of V is in H .
- For each \mathbf{u} and \mathbf{v} in H , $\mathbf{u} + \mathbf{v}$ is in H . (In this case we say H is closed under vector addition.)
- For each \mathbf{u} in H and each scalar c , $c\mathbf{u}$ is in H . (In this case we say H is closed under scalar multiplication.)

If the subset H satisfies these three properties, then H itself is a vector space.

THEOREM 1, 2 and 3 (Sections 4.1 & 4.2)

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V , then $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of V .

The null space of an $m \times n$ matrix A is a subspace of \mathbf{R}^n .

The column space of an $m \times n$ matrix A is a subspace of \mathbf{R}^m .

EXAMPLE: Determine whether each of the following sets is a vector space or provide a counterexample.

(a) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x - y = 4 \right\}$. *Solution:* Since $\begin{bmatrix} \quad \\ \quad \end{bmatrix}$ is not in H , H is not a vector space.

(b) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{l} x - y = 0 \\ y + z = 0 \end{array} \right\}$. *Solution:* Rewrite $\begin{array}{l} x - y = 0 \\ y + z = 0 \end{array}$ as

$$\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $V = \text{Nul } A$ where $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Since $\text{Nul } A$ is a subspace of \mathbf{R}^3 , V is a vector space.

$$(c) S = \left\{ \begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} : x, y, z \text{ are real} \right\}$$

One Solution: Since

$$\begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

$$S = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \right\}; \text{ therefore } S \text{ is a vector space by Theorem 1.}$$

Another Solution: Since

$$\begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

$S = \text{Col } A$ where $A = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 3 \end{bmatrix}$; therefore S is a vector space, since a column space is a vector space.

Kernal and Range of a Linear Transformation

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector \mathbf{x} in V a unique vector $T(\mathbf{x})$ in W , such that

- i. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all \mathbf{u}, \mathbf{v} in V ;
- ii. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all \mathbf{u} in V and all scalars c .

The **kernel** (or **null space**) of T is the set of all vectors \mathbf{u} in V such that $T(\mathbf{u}) = \mathbf{0}$. The **range** of T is the set of all vectors in W of the form $T(\mathbf{u})$ where \mathbf{u} is in V .

So if $T(\mathbf{x}) = A\mathbf{x}$, $\text{col } A = \text{range of } T$.