4.1 Vector Spaces & Subspaces

Many concepts concerning vectors in $\mathbf{R}^n$ can be extended to other mathematical systems.

We can think of a *vector space* in general, as a collection of objects that behave as vectors do in $\mathbf{R}^n$. The objects of such a set are called *vectors*.

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms below. The axioms must hold for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and for all scalars c and d.

1. $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $(cd)\mathbf{u} = c(d\mathbf{u})$.
10. $1\mathbf{u} = \mathbf{u}$.

Vector Space Examples

EXAMPLE: Let $M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \text{ are real} \right\}$

In this context, note that the $\mathbf{0}$ vector is $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$.

EXAMPLE: Let $n \geq 0$ be an integer and let

$$\mathbf{P}_n = \text{the set of all polynomials of degree at most } n \geq 0.$$

Members of $\mathbf{P}_n$ have the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

where $a_0, a_1, \ldots, a_n$ are real numbers and t is a real variable. The set $\mathbf{P}_n$ is a vector space.

We will just verify 3 *out of the* 10 *axioms here.*

Let $\mathbf{p}(t) = a_0 + a_1 t + \cdots + a_n t^n$ and $\mathbf{q}(t) = b_0 + b_1 t + \cdots + b_n t^n$. Let c be a scalar.

Axiom* 1*:

The polynomial $\mathbf{p} + \mathbf{q}$ is defined as follows: $(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t)$. Therefore,

$$(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t)$$

$$= (\_\_\_\_\_\_\_\_\_\_) + (\_\_\_\_\_\_\_\_\_\_)t + \cdots + (\_\_\_\_\_\_\_\_\_\_)t^n$$

which is also a ______________________ of degree at most ________. So $\mathbf{p} + \mathbf{q}$ is in $\mathbf{P}_n$.

Axiom* 4*:

$$\mathbf{0} = 0 + 0t + \cdots + 0t^n$$

(zero vector in $\mathbf{P}_n$)

$$(\mathbf{p} + \mathbf{0})(t) = \mathbf{p}(t) + \mathbf{0} = (a_0 + 0) + (a_1 + 0)t + \cdots + (a_n + 0)t^n$$

$$= a_0 + a_1 t + \cdots + a_n t^n = \mathbf{p}(t)$$

and so $\mathbf{p} + \mathbf{0} = \mathbf{p}$

Axiom* 6*:

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = (\_\_\_\_\_\_\_\_) + (\_\_\_\_\_\_\_\_)t + \cdots + (\_\_\_\_\_\_\_\_)t^n$$

which is in $\mathbf{P}_n$.

The other 7 axioms also hold, so $\mathbf{P}_n$ is a vector space.

Subspaces

Vector spaces may be formed from subsets of other vectors spaces. These are called *subspaces*.

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

b. For each **u** and **v** are in H, $\mathbf{u} + \mathbf{v}$ is in H. (In this case we say H is closed under vector addition.)

c. For each **u** in H and each scalar c, $c\mathbf{u}$ is in H. (In this case we say H is closed under scalar multiplication.)

If the subset H *satisfies these three properties, then* H *itself is a vector space.*

EXAMPLE: Let $H = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a \text{ and } b \text{ are real} \right\}$. Show that H is a subspace of $\mathbf{R}^3$.

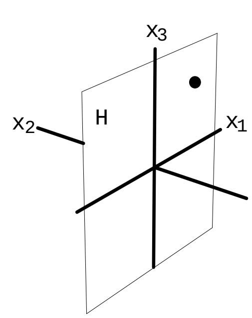
Solution: Verify properties a, b and c of the definition of a subspace.

a. The zero vector of $\mathbf{R}^3$ is in H (let $a =$ _______ and $b =$ _______).

b. Adding two vectors in H always produces another vector whose second entry is _______ and therefore the sum of two vectors in H is also in H. (H is closed under addition)

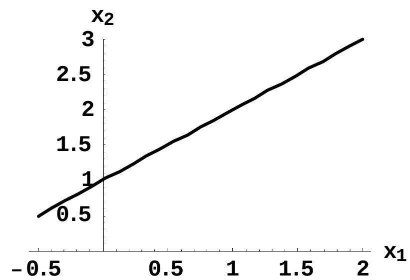
c. Multiplying a vector in H by a scalar produces another vector in H (H is closed under scalar multiplication).

Since properties a, b, and c hold, V is a subspace of $\mathbf{R}^3$. **Note:** Vectors $(a, 0, b)$ in H look and act like the points (a, b) in $\mathbf{R}^2$.



EXAMPLE: Is $H = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} : x \text{ is real} \right\}$ a subspace of _______?

I.e., does H satisfy properties a, b and c?



Graphical Depiction of H

Solution:

All three properties must hold in order for H to be a subspace of $\mathbf{R}^2$.

Property (a) is not true because

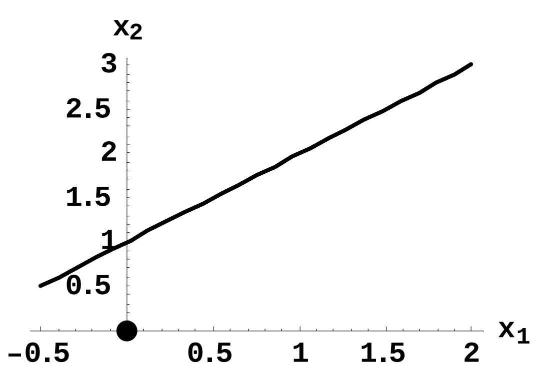
____.

Therefore H is not a subspace of $\mathbf{R}^2$.

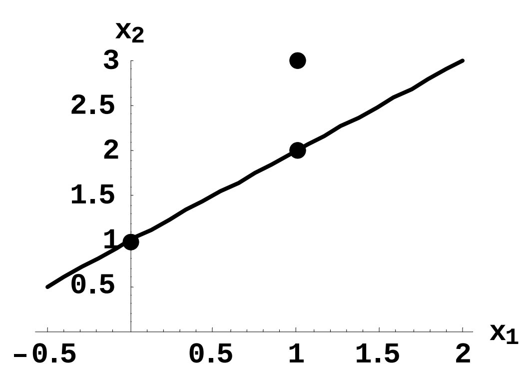
Another way to show that H is not a subspace of $\mathbf{R}^2$: Let

$$\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ then } \mathbf{u} + \mathbf{v} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

and so $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, which is _____ in H. So property (b) fails and so H is not a subspace of $\mathbf{R}^2$.



Property (a) fails



Property (b) fails

A Shortcut for Determining Subspaces

THEOREM 1

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Proof: In order to verify this, check properties a, b and c of definition of a subspace.

a. $\mathbf{0}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ since

$$\mathbf{0} = \_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\_\mathbf{v}_p$$

b. To show that Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ closed under vector addition, we choose two arbitrary vectors in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$:

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p$$

and

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$\mathbf{u} + \mathbf{v} = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p) + (b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p)$$

$$= (\_\_\_\mathbf{v}_1 + \_\_\_\_\mathbf{v}_1) + (\_\_\_\_\mathbf{v}_2 + \_\_\_\_\mathbf{v}_2) + \cdots + (\_\_\_\_\mathbf{v}_p + \_\_\_\_\mathbf{v}_p)$$

$$= (a_1 + b_1)\mathbf{v}_1 + (a_2 + b_2)\mathbf{v}_2 + \cdots + (a_p + b_p)\mathbf{v}_p.$$

So $\mathbf{u} + \mathbf{v}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

c. To show that Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ closed under scalar multiplication, choose an arbitrary number c and an arbitrary vector in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$:

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$c\mathbf{v} = c(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p)$$

$$= \_\_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\_\_\mathbf{v}_p$$

So $c\mathbf{v}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Since properties a, b and c hold, Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Recap

1. To show that H is a subspace of a vector space, use Theorem 1.
2. To show that a set is not a subspace of a vector space, provide a specific example showing that at least one of the axioms a, b or c (from the definition of a subspace) is violated.

EXAMPLE: Is $V = \{(a+2b, 2a-3b) : \ a \text{ and } b \text{ are real}\}$ a subspace of $\mathbf{R}^2$? Why or why not?
Solution: Write vectors in V in column form:

$$\begin{bmatrix} a+2b \\ 2a-3b \end{bmatrix} = \begin{bmatrix} a \\ 2a \end{bmatrix} + \begin{bmatrix} 2b \\ -3b \end{bmatrix} = \underline{\qquad}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \underline{\qquad}\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

So $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and therefore V is a subspace of _____ by Theorem 1.

EXAMPLE: Is $H = \left\{ \begin{bmatrix} a+2b \\ a+1 \\ a \end{bmatrix} : \ a \text{ and } b \text{ are real} \right\}$ a subspace of $\mathbf{R}^3$? Why or why not?

Solution: **0** is not in H since $a = b = 0$ or any other combination of values for a and b does not produce the zero vector. So property _____ fails to hold and therefore H is not a subspace of $\mathbf{R}^3$.

EXAMPLE: Is the set H of all matrices of the form $\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}$ a subspace of $M_{2\times 2}$? Explain.

Solution: Since

$$\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 3a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 3b \end{bmatrix}$$

$$= a\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} + b\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

Therefore $H = \text{Span}\left\{ \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \right\}$ and so H is a subspace of $M_{2\times 2}$.