Section 1.1: Systems of Linear Equations

A **linear equation:**

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

EXAMPLE:

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2(\sqrt{6} - x_1) + x_3$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\text{rearranged} \qquad\qquad \text{rearranged}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$3x_1 - 5x_2 = -2 \qquad\qquad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

Not linear:

$$4x_1 - 6x_2 = x_1x_2 \qquad \text{and} \qquad x_2 = 2\sqrt{x_1} - 7$$

A **system of linear equations (**or a **linear system):**
A collection of one or more linear equations involving the same set of variables, say, $x_1, x_2, \ldots, x_n$.
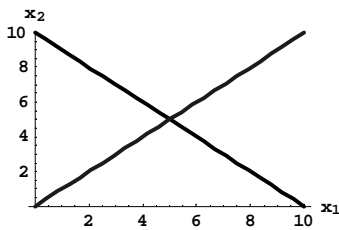
A **solution** of a linear system:
A list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.
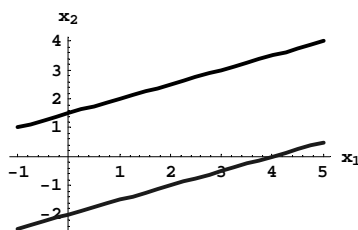
EXAMPLE Two equations in two variables:

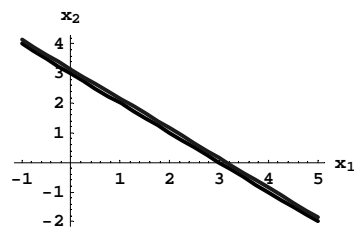$$\begin{aligned} x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0 \end{aligned} \qquad\qquad \begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8 \end{aligned} \qquad\qquad \begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$

**one unique solution**   **no solution**   **infinitely many solutions**

BASIC FACT: A system of linear equations has either
    (i) exactly one solution *(consistent)* or
    (ii) infinitely many solutions *(consistent)* or
    (iii) no solution *(inconsistent)*.

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

i) The planes intersect in one point. *(one solution)*

ii) The planes intersect in one line. *(infinitely many solutions)*

iii) There is not point in common to all three planes. *(no solution)*
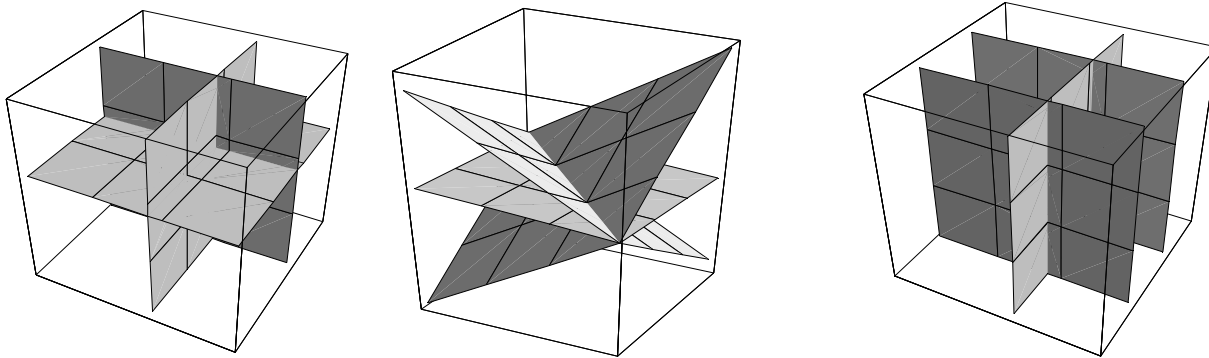
The **solution set:**
- The set of all possible solutions of a linear system.
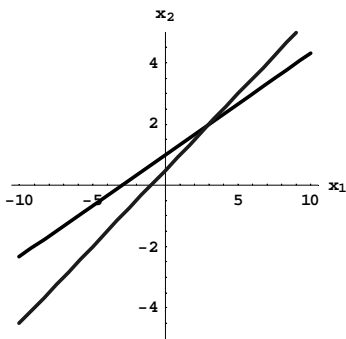
Equivalent systems:
- Two linear systems with the same solution set.
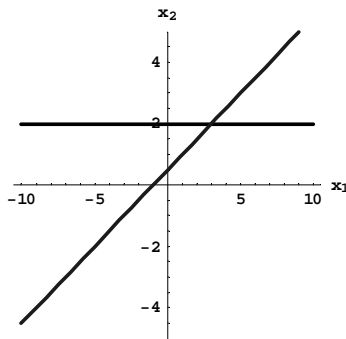
STRATEGY FOR SOLVING A SYSTEM:
- *Replace one system with an equivalent system that is easier to solve.*

EXAMPLE:

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \quad \rightarrow \quad \begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned} \quad \rightarrow \quad \begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \qquad \begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned} \qquad \begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned}$$
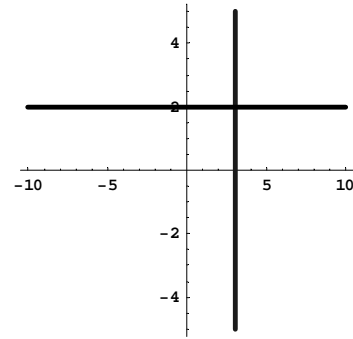
Matrix Notation

$$x_1 \ - \ 2x_2 \ = \ -1 \qquad \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$
$$-x_1 \ + \ 3x_2 \ = \ 3$$

(coefficient matrix)

$$x_1 \ - \ 2x_2 \ = \ -1 \qquad \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \end{bmatrix} \rightarrow \qquad x_1 \ - \ 2x_2 \ = \ -1 \qquad \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$
$$-x_1 \ + \ 3x_2 \ = \ 3 \qquad\qquad\qquad\qquad\qquad x_2 \ = \ 2$$

(augmented matrix)

$$\downarrow$$

$$x_1 \qquad\quad = \ 3 \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$
$$x_2 \ = \ 2$$

Elementary Row Operations:
1. *(Replacement)* Add one row to a multiple of another row.
2. *(Interchange)* Interchange two rows.
3. *(Scaling)* Multiply all entries in a row by a nonzero constant.

Row equivalent matrices: Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Fact about Row Equivalence: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

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EXAMPLE:

$$
\begin{array}{rcl}
x_1 - 2x_2 + x_3 &=& 0 \\
2x_2 - 8x_3 &=& 8 \\
-4x_1 + 5x_2 + 9x_3 &=& -9
\end{array}
\qquad
\left[\begin{array}{cccc}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
-4 & 5 & 9 & -9
\end{array}\right]
$$

$$
\begin{array}{rcl}
x_1 - 2x_2 + x_3 &=& 0 \\
2x_2 - 8x_3 &=& 8 \\
-3x_2 + 13x_3 &=& -9
\end{array}
\qquad
\left[\begin{array}{cccc}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
0 & -3 & 13 & -9
\end{array}\right]
$$

$$
\begin{array}{rcl}
x_1 - 2x_2 + x_3 &=& 0 \\
x_2 - 4x_3 &=& 4 \\
-3x_2 + 13x_3 &=& -9
\end{array}
\qquad
\left[\begin{array}{cccc}
1 & -2 & 1 & 0 \\
0 & 1 & -4 & 4 \\
0 & -3 & 13 & -9
\end{array}\right]
$$

$$
\begin{array}{rcl}
x_1 - 2x_2 + x_3 &=& 0 \\
x_2 - 4x_3 &=& 4 \\
x_3 &=& 3
\end{array}
\qquad
\left[\begin{array}{cccc}
1 & -2 & 1 & 0 \\
0 & 1 & -4 & 4 \\
0 & 0 & 1 & 3
\end{array}\right]
$$

$$
\begin{array}{rcl}
x_1 - 2x_2 &=& -3 \\
x_2 &=& 16 \\
x_3 &=& 3
\end{array}
\qquad
\left[\begin{array}{cccc}
1 & -2 & 0 & -3 \\
0 & 1 & 0 & 16 \\
0 & 0 & 1 & 3
\end{array}\right]
$$

$$
\begin{array}{rcl}
x_1 &=& 29 \\
x_2 &=& 16 \\
x_3 &=& 3
\end{array}
\qquad
\left[\begin{array}{cccc}
1 & 0 & 0 & 29 \\
0 & 1 & 0 & 16 \\
0 & 0 & 1 & 3
\end{array}\right]
$$

Solution: $(29, 16, 3)$

Check: Is $(29, 16, 3)$ a solution of the *original* system?

$$
\begin{array}{rcl}
x_1 - 2x_2 + x_3 &=& 0 \\
2x_2 - 8x_3 &=& 8 \\
-4x_1 + 5x_2 + 9x_3 &=& -9
\end{array}
$$

$$
\begin{array}{rcl}
(29) - 2(16) + 3 &= 29 - 32 + 3 &= 0 \\
2(16) - 8(3) &= 32 - 24 &= 8 \\
-4(29) + 5(16) + 9(3) &= -116 + 80 + 27 &= -9
\end{array}
$$

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Two Fundamental Questions (Existence and Uniqueness)

1) Is the system consistent; (i.e. does a solution **exist**?)

2) If a solution exists, is it **unique**? (i.e. is there one & only one solution?)

EXAMPLE: Is this system consistent?

$$
\begin{array}{rcrcrcr}
x_1 & - & 2x_2 & + & x_3 & = & 0 \\
 & & 2x_2 & - & 8x_3 & = & 8 \\
-4x_1 & + & 5x_2 & + & 9x_3 & = & -9
\end{array}
$$

In the last example, this system was reduced to the triangular form:

$$
\begin{array}{rcrcrcr}
x_1 & - & 2x_2 & + & x_3 & = & 0 \\
 & & x_2 & - & 4x_3 & = & 4 \\
 & & & & x_3 & = & 3
\end{array}
\qquad
\left[
\begin{array}{cccc}
1 & -2 & 1 & 0 \\
0 & 1 & -4 & 4 \\
0 & 0 & 1 & 3
\end{array}
\right]
$$

This is sufficient to see that the system is consistent and unique. Why?

EXAMPLE: Is this system consistent?

$$
\begin{aligned}
3x_2 - 6x_3 &= 8 \\
x_1 - 2x_2 + 3x_3 &= -1 \\
5x_1 - 7x_2 + 9x_3 &= 0
\end{aligned}
\qquad
\begin{bmatrix}
0 & 3 & -6 & 8 \\
1 & -2 & 3 & -1 \\
5 & -7 & 9 & 0
\end{bmatrix}
$$

Solution: Row operations produce:

$$
\begin{bmatrix}
0 & 3 & -6 & 8 \\
1 & -2 & 3 & -1 \\
5 & -7 & 9 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -2 & 3 & -1 \\
0 & 3 & -6 & 8 \\
0 & 3 & -6 & 5
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -2 & 3 & -1 \\
0 & 3 & -6 & 8 \\
0 & 0 & 0 & -3
\end{bmatrix}
$$

Equation notation of triangular form:

$$
\begin{aligned}
x_1 - 2x_2 + 3x_3 &= -1 \\
3x_2 - 6x_3 &= 8 \\
0x_3 &= -3 \qquad \leftarrow \textit{Never true}
\end{aligned}
$$

The original system is inconsistent!

EXAMPLE: For what values of h will the following system be consistent?

$$
\begin{aligned}
3x_1 - 9x_2 &= 4 \\
-2x_1 + 6x_2 &= h
\end{aligned}
$$

Solution: Reduce to triangular form.

$$
\begin{bmatrix}
3 & -9 & 4 \\
-2 & 6 & h
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -3 & \frac{4}{3} \\
-2 & 6 & h
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -3 & \frac{4}{3} \\
0 & 0 & h + \frac{8}{3}
\end{bmatrix}
$$

The second equation is $0x_1 + 0x_2 = h + \frac{8}{3}$. System is consistent only if $h + \frac{8}{3} = 0$ or $h = \frac{-8}{3}$.