Chapter 1

System of Linear Equations

$$\begin{array}{ccc} a_{11}x_1 + \cdots & + a_{1n}x_n = & b_1 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + & + a_{mn}x_n = & b_m \end{array}$$

$$\Updownarrow$$

Vector Equation

$$x_1 \begin{bmatrix} a_{11} \\ \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \\ b_m \end{bmatrix}$$

$$\Updownarrow$$

Matrix Equation

$$\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix}}_{= A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{= x} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{= b}$$

Solve such systems by row reducing the augmented matrix $[A\,|\,b]$ to reduced echelon form.

Echelon Forms

Echelon Form

1. All nonzero rows are above any rows of all zeroes.

2. Each leading entry of a row is in a column to the right of the leading entry in the row above it.

3. All entries in a column below the leading entry are 0.

$$\begin{bmatrix} \square & * & * & \cdots & \\ & \square & \cdots & \cdots & \\ & & \square & \cdots & \cdots \\ 0 & 0 & - & - & 0 \\ 0 & 0 & - & - & 0 \end{bmatrix}$$

Reduced Echelon Form

4. The leading entry (pivot) in each nonzero row is <u>1</u>

5. Each leading 1 is the only non-zero entry in its column.

<u>Thm1</u> The reduced echelon form of a matrix is unique - ie each matrix is row equivalent to one and only one reduced echelon matrix.

The standard (non-reduced) echelon form is not unique.

<u>Thm2</u> (P.24) A linear system $Ax = b$ is consisistent iff the rightmost column of the reduced echelon form (r.e.f.) of the augmented matrix $[A \mid b]$ is <u>not</u> a pivot column.

Matrix Equations

$Ax = 0$ has a non-trivial sol^{n} iff there are free variables

$Ax = b$ has a sol^{n} iff b is a linear combination of the columns of A.

Theorem 4 (P.43) A $m \times n$ matrix. TFAE

a). $\forall b \in \mathbb{R}^n$, $Ax = b$ has a sol^{n}.

b). Each b in $\mathbb{R}^m$ is a lin. comb. of the cols. of A.

c). The cols. of A span $\mathbb{R}^m$.

d). A has a pivot position in every row.

The general solution to $Ax = b$ is obtained by adding one solution to $Ax = b$ to the general solution to $Ax = 0$. (Theorem 6, P. 53)

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear

if $\begin{aligned} T(cv) &= cT(v) \\ T(u+v) &= T(u) + T(v) \end{aligned}$, $\forall$ vectors u, v $\forall$ scalars c.

The standard matrix of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is

$$A = \begin{bmatrix} | & & | \\ T(e_1) & \cdots & T(e_n) \\ | & & | \end{bmatrix}$$

where $\{e_1, \ldots, e_n\}$ is the std. basis for $\mathbb{R}^n$.

Chapter 2 Matrices

Inverses (only for square matrices) $AA^{-1} = A^{-1}A = I_n$.

To find the inverse of an $n \times n$ matrix A row reduce the augmented matrix

$$[A \mid I_n]$$

to

$$[I_n \mid A^{-1}]$$

ie row reduce the lhs to I_n & the rhs is then automatically A^{-1}!

Thm 6. A, B inv. $n \times n$ matrices

a. A^{-1} inv & $(A^{-1})^{-1} = A$

b. AB inv. & $(AB)^{-1} = B^{-1}A^{-1}$

c. A^T inv & $(A^T)^{-1} = (A^{-1})^T$.

<u>Theorem 8</u> Invertible Matrix Theorem (P.129).

A $n \times n$. matrix. TFAE

a. A inv. ie $\exists$ B s.t. $AB = BA = I_n$.

b. A is row equivalent to I_n.

c. A has n pivot positions

d. $Ax = 0$ has only the trivial sol$^{\Delta}$ $x = 0$.

e. The cols. of A are lin ind.

f. The linear transformation $x \longmapsto Ax$ is one-to-one.

g. $Ax = b$ has at least one sol$^{\Delta}$ $\forall b \in \mathbb{R}^n$.

h. The cols. of A span $\mathbb{R}^n$

i. The linear transformation $x \longmapsto Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$

j. $\exists$ an $n \times n$ matrix C st. $CA = I_n$

k. $\exists$ an $n \times n$ matrix D s.t. $AD = I_n$.

l. A^T is inv.

p.t.o.

m. The cols. of A form a basis of $\mathbb{R}^n$.

n. $\operatorname{Col} A = \mathbb{R}^n$.

o. $\dim \operatorname{Col} A = n$.

p. $\operatorname{Rank} A = n$.

q. $\operatorname{Nul} A = \{0\}$.

r. $\dim \operatorname{Nul} A = 0$.

s. 0 is not an eigenvalue of A.

t. $\det A \neq 0$.

Chapter 3 Determinants

A $n \times n$ matrix

A_{ij} minor matrix $(n-1) \times (n-1)$ obtained by blocking out the i-th row and j-th column of A.

$$\det A = (-1)^{i+1} \det A_{i1} + (-1)^{i+2} \det A_{i2} + \cdots + (-1)^{i+n} \det A_{in}$$

- cofactor expansion along row i

$$= (-1)^{1+j} \det A_{1j} + (-1)^{2+j} \det A_{2j} + \cdots + (-1)^{n+j} \det A_{nj}$$

- cofactor expansion along col. j.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad 2 \times 2$$

$\det A = ad - bc.$

If A is upper or lower triangular, then $\det A$ is the product of the diagonal entries.

Cofactor expansion is computationally intensive. The best idea is to use row operations to make A as nearly triangular as possible before computing $\det A$.

<u>Remark</u> Elementary row operations leave $\det A$ unchanged <u>except</u> for swapping rows which changes the sign of $\det A$. and multiplying <u>one</u> row by a scalar factor which changes $\det A$ by this factor.

<u>Facts</u>.

$$\det(AB) = \det A \det B$$

$$\det A^T = \det A$$

$$A \text{ inv} \Rightarrow \det A^{-1} = \frac{1}{\det A}.$$

Chapter 4 Vector Spaces

Subspaces A subset H of a vector space V is a subspace of V if

1. $0_V \in H$
2. H is closed under addition - ie. $u+v \in H \quad \forall u, v \in H$
3. H is closed under scalar multiplication - ie $cu \in H \quad \forall u \in H, c \in \mathbb{R}$.

(Defn of subspace).

Linear Independence - Bases

Let V be a v.s. & $v_1, \ldots, v_p \in V$.

$\text{Span}\{v_1, \ldots, v_p\} = \{\text{all linear combs. of } v_1, \ldots, v_p\}$

$= \{c_1v_1 + \ldots + c_pv_p, \ c_1, \ldots, c_n \in \mathbb{R}\}$.

$\text{Span}\{v_1, \ldots, v_p\}$ is a subspace of V.

If $\text{Span}\{v_1, \ldots, v_p\} = V$, we say $\{v_1, \ldots, v_p\}$ spans V.

$\{v_1, \ldots, v_p\}$ is linearly independent if

$$c_1v_1 + \ldots + c_pv_p = 0 \Rightarrow c_1 = c_2 = \ldots = c_p = 0$$

(only trivial solution).

If $\{v_1, \ldots, v_p\}$ spans V & is lin. ind., we say it is a <u>basis</u> for V.

Simple cases

<u>1 vector</u> $\{v\}$ is lin. ind. iff $v \neq 0$.

<u>2 vectors</u> $\{v, w\}$ is lin ind. iff neither vector is a scalar multiple of the other.

Theorem 5 <u>Spanning Set Theorem</u> p. 239.

Let $S = \{v_1, \_\_, v_p\}$ be a set in V and let $H = \operatorname{Span}\{v_1, \_\_, v_p\}$.

a. If one of the vectors in S – say v_k is a lin. comb. of the remaining vectors in S, then the set formed from S by removing v_k still spans H.

b. If $H \neq \{0\}$, some subset of S is a basis for H.

Can use this to make an algorithm to extract a basis from S. Usually quicker to do it by making the vectors the columns of a matrix A and then finding col A.

Dimension.

If a v.-s. V is spanned by a finite set, we say say V is finite-dimensional. In this case the number of elements in any basis of V is always the same (Theorem 10, p. 257) and is called the dimension of V.

Nulspace, Column Space

A $m \times n$ matrix

$x \longmapsto Ax$ gives a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

$\text{Nul } A = \{x : Ax = 0\}$ is a subspace of $\mathbb{R}^n$.

$\text{Col } A = \text{span}\{\text{columns of } A\}$

$= \{b : Ax = b \text{ for some } x \in \mathbb{R}^n\}$

is a subspace of $\mathbb{R}^m$.

To find Nul A, obtain the r.e.f. of $[A \mid 0]$ and write the solution in parametric vector form in terms of the free variables. The vectors multiplying each free variable give a basis for Nul A and the dimension of Nul A is the number of free variables.

To find Col A, obtain the echelon form of A (no need for r.e.f) to determine the pivot columns of A. The pivot columns of A (<u>not</u> the reduced matrix!) give a basis for Col A.

Thm 14 Rank Thm

Rank A := dim Col A.

$$\dim \text{Nul } A + \text{Rank } A = \# \text{ free variables} + \# \text{ basic variables}$$

$$= n \quad (\text{total } \# \text{ of variables})$$

Row A = span{Rows of A}

dim Row A = dim Col A = Rank A.

Coordinates, Change of Basis

$\mathcal{B} = \{b_1, \ , b_n\}$ basis for a v.s. V.

For $x \in V$, if $x = c_1 v_1 + \cdots + c_n v_n$, then the numbers $c_1, \_\_, c_n$ are the coordinates of x with respect to the basis $\mathcal{B}$.

Coordinate Mapping
$$\underset{\substack{\cap \\ V}}{x} \longmapsto \underset{\substack{\cap \\ \mathbb{R}^n}}{\begin{bmatrix} c_1 \\ | \\ | \\ c_n \end{bmatrix}}$$
written as $[x]_{\mathcal{B}}$

If $V = \mathbb{R}^n$, then the matrix

$$P_{\mathcal{B}} = [b_1 \cdots b_n]$$

is the change of coordinates matrix from $\mathcal{B}$ to the std. basis in $\mathbb{R}^n$.

$$x = P_{\mathcal{B}} [x]_{\mathcal{B}}$$

If we have two bases

$\mathcal{B} = \{b_1, \_\_, b_n\}$, $\mathcal{C} = \{c_1, \_\_, c_n\}$ for V,

then the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$, $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ is given by

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[[b_1]_{\mathcal{C}} - - [b_n]_{\mathcal{C}} \right]$$

coords of the b_i's in $\mathcal{C}$.

Have

$$[x]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [x]_{\mathcal{B}}$$

To find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$, either reduce

$[c_1 \ldots c_n \mid b_1 - - b_n]$ to $[I_n \mid \underset{\mathcal{C} \leftarrow \mathcal{B}}{P}]$ or

calculate $P_{\mathcal{C}}^{-1} P_{\mathcal{B}}$ which gives the same result.

<u>Note</u> $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \left[\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} \right]^{-1}$.

Chapter 5 Eigenvalues and Eigenvectors

A $n \times n$ matrix.

If $Ax = \lambda x$ has a sol^n for some $x \neq 0$ and scalar λ, we say x is an eigenvector of A with eigenvalue λ.

Facts.

The eigenvalues of a triangular matrix are its diagonal entries (Theorem 1, P. 306).

If $v_1, \ldots, v_r$ are e-vectors corresponding to distinct e-values $\lambda_1, \ldots, \lambda_r$, then $\{v_1, \ldots, v_r\}$ is lin. ind. (Theorem 2, P. 307).

Given an e-value λ, solve

$(A - \lambda I)x = 0$ to find the

eigenspace associated with λ.

To find the e-values, solve the
characteristic equation

$$\det(A - \lambda I_n) = 0$$

which is a polynomial of degree n
whose roots are the eigenvalues.

Diagonalization

A is diagonal if $\exists$ an inv. matrix P and a diagonal matrix D s.t.

$$A = PDP^{-1}$$

In this case $AP = PD$ and the cols. of P are (must be) e-vectors of A. Say A is diagonalizable.

Fact

A is diagonalizable $\Longleftrightarrow$ A has n lin. ind. e-vectors.

Not every matrix is diagonalizable!

Chapter 6 Inner Product, Length, Orthogonality

$u, v \in \mathbb{R}^n$ are orthogonal if $u \cdot v = 0$

Pythagorean Theorem (P. 380) $u \cdot v = 0$ iff $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

Orthogonal Complements

If W is a subspace of $\mathbb{R}^n$,

$$W^\perp = \{ v : v \cdot w = 0 \ \forall w \in W \}$$

is another subspace of $\mathbb{R}^n$, the orthogonal complement. We have

$$\dim W + \dim W^\perp = n \ (= \dim \mathbb{R}^n).$$

Theorem 3 A $m \times n$ matrix.

$$(\text{Row } A)^\perp = \text{Nul } A, \quad (\text{Col } A)^\perp = \text{Nul}(A^T).$$

Angles $\cos \theta = \dfrac{u \cdot v}{\|u\| \|v\|}$

$S = \{u_1, \ldots, u_p\}$ is an orthogonal set if $u_i \cdot u_j = 0 \quad \forall\, i \neq j$

Orthogonal sets of nonzero vectors are lin ind. (Thm 4, p. 384).

Thm 5 If $\{u_1, \ldots, u_p\}$ is an orthogonal basis for some subspace W of $\mathbb{R}^n$ and $y \in W$ so that

$$y = c_1 u_1 + \quad + c_p u_p$$

for some $c_1, \quad , c_p$, then

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j}, \quad 1 \leq j \leq p.$$

Orthogonal Projection

W as before, y any vector in $\mathbb{R}^n$.

Then $y = \hat{y} + z$

where $z \in W^{\perp}$ & $\hat{y} \in W$ with

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \text{——} + \frac{y \cdot u_p}{u_p \cdot u_p} u_p$$

$\Big($ $= (y \cdot u_1) u_1 + \qquad + (y \cdot u_p) u_p$

if we have an orthonormal basis $\Big)$

$\hat{y}$ is the closest vector in W to y,

ie $\|y - \hat{y}\| \leq \|y - v\| \quad \forall v \in W.$

(Theorem 9, p. 398).

Gram Schmidt Process (P. 404).

Given a basis $\{x_1, \ldots, x_p\}$ for a subspace W of $\mathbb{R}^n$, define

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$\vdots$$

$$v_p = x_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_p \cdot v_2}{v_2 \cdot v_2} v_2 - \cdots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1},$$

Then $\{v_1, \ldots, v_p\}$ is an orthogonal basis for W and

$$\text{Span}\{v_1, \ldots, v_k\} = \text{Span}\{x_1, \ldots, x_k\} \quad 1 \le k \le p.$$