1. (a) $\sum_{n=0}^{\infty} \frac{2^n \cdot x^{2n}}{(2n)!}$

$$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{\left| \frac{2^{n+1} \cdot x^{2(n+1)}}{(2(n+1))!} \right|}{\left| \frac{2^n \cdot x^{2n}}{(2n)!} \right|} = \lim_{n \to \infty} \frac{2 \cdot x^2}{(2n+2)(2n+1)} = 0$$

So the radius of convergence is infinite.

(b)
$$\sum_{n=0}^{\infty} \frac{4^n \cdot (n!)^2 \cdot (x-1)^n}{(2n)!}$$

$$\lim_{n \to \infty} \frac{\left| a_{n+1} \right|}{\left| a_n \right|} = \lim_{n \to \infty} \frac{\left| \frac{4^{n+1} \cdot \left((n+1)! \right)^2 \cdot (x-1)^{n+1}}{\left(2(n+1) \right)!} \right|}{\left| \frac{4^n \cdot (n!)^2 \cdot (x-1)^n}{(2n)!} \right|} = \lim_{n \to \infty} \frac{4 \cdot (n+1)^2 \cdot |x-1|}{(2n+2)(2n+1)} = 1 \cdot |x-1|$$

The radius of convergence is 1.

2. (a) For n = 4, a = 0 and $f(x) = \cos x$,

$$f(a) = \cos a = \cos 0 = 1$$

$$f'(a) = -\sin a = -\sin 0 = 0$$

$$f''(a) = -\cos a = -\cos 0 = -1$$

$$f'''(a) = \sin a = \sin 0 = 0$$

$$f^{(4)}(a) = \cos a = \cos 0 = 1$$

$$f(x) = \frac{f(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

Thus, for x near 0,

$$\cos x \approx \frac{1}{0!} \cdot 1 + \frac{0}{1!} \cdot (x - 0)^1 + \frac{-1}{2!} \cdot (x - 0)^2 + \frac{0}{3!} \cdot (x - 0)^3 + \frac{1}{4!} \cdot (x - 0)^4$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

(b) For
$$x$$
 near 1, $e^x \approx e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \frac{e}{4!}(x-1)^4$

(c) For
$$n = 4$$
, $a = 0$, $f(x) = \sqrt[3]{1-x}$

$$f(a) = (1-a)^{\frac{1}{3}} = 1$$

$$f'(a) = -\frac{1}{3}(1-a)^{-\frac{2}{3}} = -\frac{1}{3}$$

$$f''(a) = -\frac{2}{9}(1-a)^{-\frac{5}{3}} = -\frac{2}{9}$$

$$f'''(a) = -\frac{10}{27}(1-a)^{-\frac{8}{3}} = -\frac{10}{27}$$

$$f^{(4)}(a) = -\frac{80}{81}(1-a)^{-\frac{11}{3}} = -\frac{80}{81}$$

$$f(x) \approx 1 - \frac{1}{3}x - \frac{\frac{2}{9}x^2}{2!} - \frac{\frac{10}{27}x^3}{3!} - \frac{\frac{80}{81}x^4}{4!}$$

$$= 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{9}x^3 - \frac{10}{243}x^4$$

(d) For
$$x$$
 near $\frac{\pi}{2}$, $\sin x \approx 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!}$.

(e) For x near 1,
$$\ln(x^2) \approx 0 + 2(x-1) - \frac{2(x-1)^2}{2!} + \frac{4(x-1)^3}{3!}$$
.

3. (a)
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{\left(-1\right)^{n+1} \cdot x^{2n-1}}{\left(2n-1\right)!} + \dots$$
$$\sin x = \sum_{n=1}^{\infty} \frac{\left(-1\right)^{n+1} \cdot x^{2n-1}}{\left(2n-1\right)!}$$

(b)
$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + \frac{(-1)^{n+1} \cdot x^{2n-1}}{2n-1} + \dots$$

 $\arctan x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot x^{2n-1}}{2n-1}$

(c)
$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \dots + \frac{(-1)^n \cdot x^{2n}}{n!} + \dots$$

 $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{n!}$

4. Given: y is near 0. Change the problem slightly so that (c) is $\frac{1}{\sqrt{1-y^2}}-1$.

$$\ln\left(1+\frac{y^2}{2}\right) \approx \frac{\frac{y^2}{2}}{1} - \frac{\left(\frac{y^2}{2}\right)^2}{2} + \frac{\left(\frac{y^2}{2}\right)^3}{3} - \frac{\left(\frac{y^2}{2}\right)^4}{4} = \frac{1}{2}y^2 - \frac{1}{8}y^4 + \frac{1}{24}y^6 - \frac{1}{64}y^8$$

$$1 - \cos y \approx 1 - \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \frac{y^8}{8!}\right) = \frac{1}{2}y^2 - \frac{1}{24}y^4 + \frac{1}{720}y^6 - \frac{1}{40,320}y^8$$

$$\frac{1}{\sqrt{1-y^2}} = \left(1 + \left(-y^2\right)\right)^{-\frac{1}{2}}, \text{ so use the binomial series to get: } \left(1 + \left(-y^2\right)\right)^{-\frac{1}{2}} \approx \frac{1}{\sqrt{1-y^2}} = \left(1 + \left(-y^2\right)\right)^{-\frac{1}{2}}, \text{ so use the binomial series to get: } \left(1 + \left(-y^2\right)\right)^{-\frac{1}{2}} \approx \frac{1}{\sqrt{1-y^2}} = \frac{$$

$$1 + \left(-\frac{1}{2}\right)\left(-y^{2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(-y^{2}\right)^{2} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\left(-y^{2}\right)^{3} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)}{4!}\left(-y^{2}\right)^{2}$$

$$= 1 + \frac{1}{2}y^{2} + \frac{3}{8}y^{4} + \frac{5}{16}y^{6} + \frac{35}{128}y^{8}. \text{ So, } \frac{1}{\sqrt{1-y^{2}}} - 1 \approx \frac{1}{2}y^{2} + \frac{3}{8}y^{4} + \frac{5}{16}y^{6} + \frac{35}{128}y^{8}.$$

By examining the polynomial approximations, one can see that they all differ in the 2^{nd} term. Since y is near 0, we can ignore all of the terms beyond the 2^{nd} term.

Thus,
$$\ln\left(1+\frac{y^2}{2}\right) < 1-\cos y < \frac{1}{\sqrt{1-y^2}} - 1$$
. (a

5. Examine $f(x) = \cos x$ on the interval [0,1] with a = 0.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

Since we only have even powers, assume n is even and write

$$\cos x = x^{0} - \frac{x^{2}}{2!} + \dots + (-1)^{\frac{n}{2}} \frac{x^{n}}{n!} + E_{n}(x)$$

$$|E_n(x)| = |f(x) - P_n(x)| \le \frac{M}{(n+1)!} |x|^{n+1}$$

where $\max |f^{(n+1)}(x)| \le M$ on the interval between 0 and x.

Since $\pm \cos x$ and $\pm \sin x$ are both ≤ 1 on the interval [0,1], we can let M = 1.

Thus,
$$|E_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$$
.

(a) To estimate cos 1, $|E_n(1)| \le \frac{1}{(n+1)!}$. The phrase "accurate to at least 4

decimal places", means that the error is $\leq 10^{-4}$. So, set $\frac{1}{(n+1)!} \leq 10^{-4}$.

This means that we must have $(n+1)! \ge 10^4$.

Since $7! < 10^4 < 8!$, n + 1 must be at least 8, so n = 7.

Thus, $\cos 1 \approx 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} \approx 0.5402777778$.

(b) If we want to be accurate to at least 6 decimal places, then we want to make sure that $|E_n(1)| \le 10^{-6}$.

Which means that we must have $(n+1)! \ge 10^6$.

Since $9! < 10^6 < 10!$, n + 1 must be at least 10, so n = 9.

Thus, $\cos 1 \approx 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} \approx 0.5403025794$.

6. Show $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots$

1st: Show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for any real number x.

(In other words, begin with the Taylor polynomial centered at zero and show that its radius of convergence is infinite)

$$\lim_{n\to\infty} \frac{\left|a_{n+1}\right|}{\left|a_n\right|} = \lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{(n+1)!}\right|}{\left|\frac{x^n}{n!}\right|} = \lim_{n\to\infty} \frac{\left|x\right|}{n+1} = 0 \text{ for any real number } x.$$

So the radius of convergence is infinity, so the series converges for all x.

2nd: Show that the infinite series converges to e^x .

The error bound tells us how close the finite series is to the function e^x . So in order for the infinite series to converge to e^x , the error must converge to zero as $n \to \infty$.

$$|E_n(x)| = |e^x - P_n(x)| \le \frac{M}{(n+1)!} |x|^{n+1}$$

where $\max |f^{(n+1)}(x)| \le M$ for all real numbers.

(Note: $f^{(n+1)}$ means the $(n+1)^{st}$ derivative of f)

Since $f^{(n)}(x) = e^x$ for any n, we can let $M = e^x$.

Thus,
$$|E_n(x)| \le \frac{e^x \cdot |x|^{n+1}}{(n+1)!}$$
.

Since
$$\lim_{n\to\infty} \frac{e^x \cdot |x|^{n+1}}{(n+1)!} = e^x \cdot \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = e^x \cdot 0 = 0$$
, (See bottom of page 500)

the error converges to zero. Which means that the series converges to e^x .