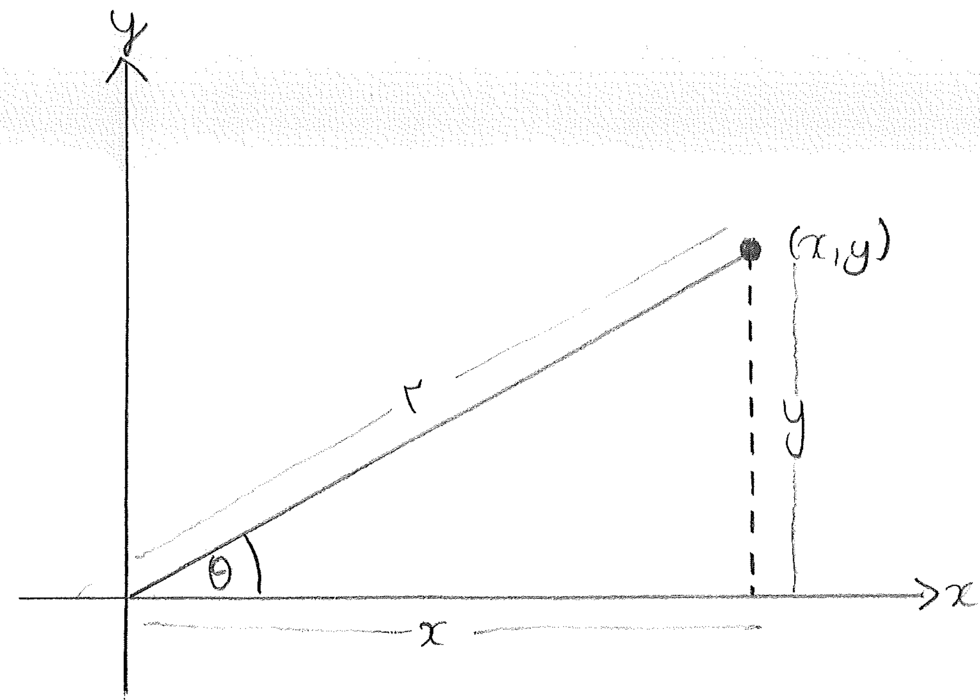
§ 8.3 Polar Coordinates

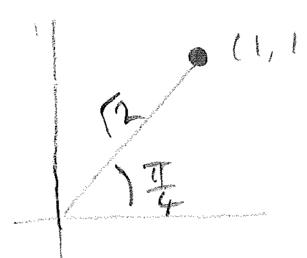


Conversion of polar coords. to Cartesian coords.

$$\frac{x}{r} = \cos\theta \ , \quad \frac{y}{r} = \sin\theta$$

So $\boxed{x = r\cos\theta, \ y = r\sin\theta}$

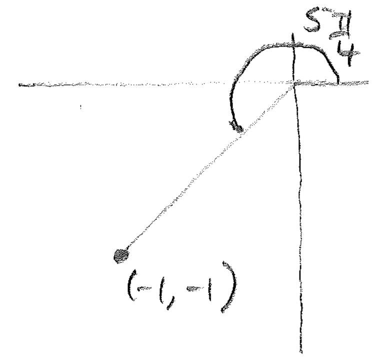
e.g. If $\theta = \frac{\pi}{4}$, $r = \sqrt{2}$



$$x = \sqrt{2}\cdot\cos\frac{\pi}{4} = \sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1$$

$$y = \sqrt{2}\sin\frac{\pi}{4} = \sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1$$

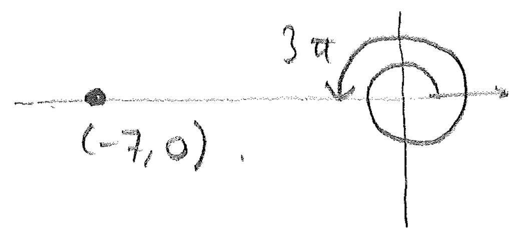
If $\theta = \frac{5\pi}{4}$, $r = \sqrt{2}$



$x = \sqrt{2}\cos\left(\frac{5\pi}{4}\right) = \sqrt{2} \cdot \frac{-1}{\sqrt{2}} = -1$

$y = \sqrt{2}\sin\left(\frac{5\pi}{4}\right) = \sqrt{2} \cdot \frac{-1}{\sqrt{2}} = -1$

If $\theta = 3\pi$, $r = 7$



$x = 7\cos(3\pi) = 7(-1) = -7$

$y = 7\sin(3\pi) = 7(0) = 0$

Conversion of Cartesian Coords to Polar Coords.

$\frac{y}{x} = \tan\theta,\ x \neq 0$ $\qquad x^2 + y^2 = r^2$

$\tan\theta = \frac{y}{x},\ x \neq 0$ $\qquad r^2 = x^2 + y^2$

So

$$\boxed{\theta = \arctan\frac{y}{x}, \quad r = \sqrt{x^2+y^2}}$$

Note that $\theta = \arctan\left(\frac{y}{x}\right)$ is not sufficient to tell in which quadrant θ should be.

e.g. If $x = 1, y = 1$

$\theta = \arctan\left(\frac{1}{1}\right) = \arctan 1 = \frac{\pi}{4}$ as we need a pt in the first quadrant

However, if $x = -1, y = -1$

$\theta = \arctan\left(\frac{-1}{-1}\right) = \arctan 1 = \frac{5\pi}{4}$ as we need a pt. in the third quadrant

Note that there are many choices of θ for a given (x, y).

e.g. If $x = -7, y = 0$

$\theta = \arctan(0)$

$\theta = \pi$ will do as it's on the $-$ve x-axis

However, so would $\theta = -\pi$, $\theta = 3\pi$, $\theta = 5\pi$ etc.

In fact, all possible values of θ are

$\theta = \pi + 2n\pi$, where n is <u>any</u> integer

What if $x = 0$?

Here $\theta = \arctan\left(\frac{y}{x}\right)$ doesn't make sense.

However, if $y > 0$, then we need a value of θ which puts us on the positive y-axis (e.g $\theta = \frac{\pi}{2}$, $\theta = \frac{5\pi}{2}$).

If $y < 0$, then we need a value of θ which puts us on the negative y-axis (e.g. $\theta = -\frac{\pi}{2}$, $\theta = \frac{3\pi}{2}$).

Finally if $x = y = 0$ and we're at the origin, then $r = 0$ and θ is <u>undefined</u>.

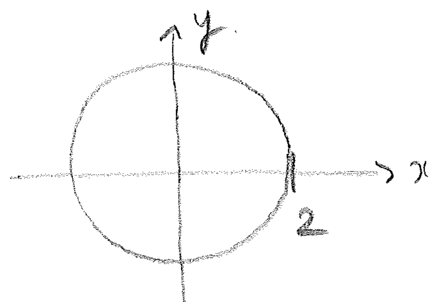
Except for this case, given (x, y) all possible values of θ differ by integer multiples of 2π like in the example above.

Curves in Polar Coordinates

i) $r = 2$

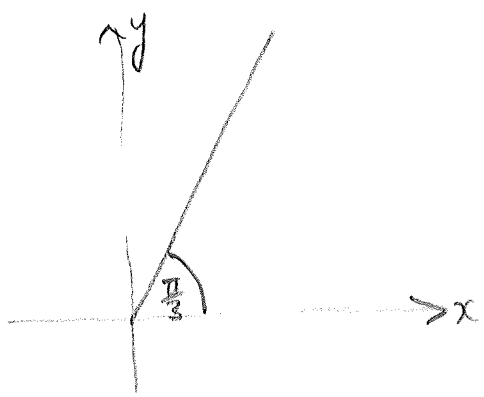
This is just a circle of radius 2 about the origin since

$r = 2 \Rightarrow \sqrt{x^2+y^2} = 2 \qquad x^2 + y^2 = 2^2$



ii) $\theta = \frac{\pi}{3}$

This is a half-line with slope $\tan\frac{\pi}{3} = \sqrt{3}$ which extends from the origin.

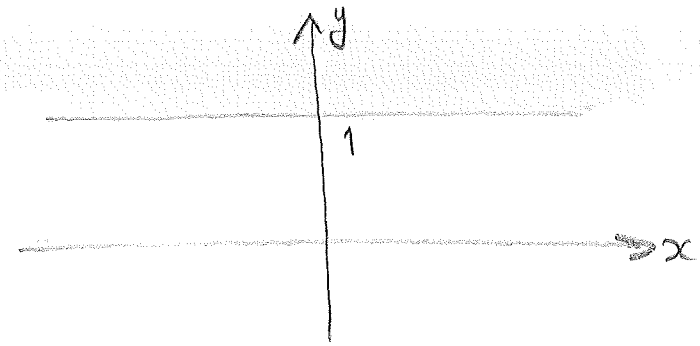


iii) $y = 1$.

This is a horizontal line in cartesian coordinates. Since $y = r\sin\theta$,

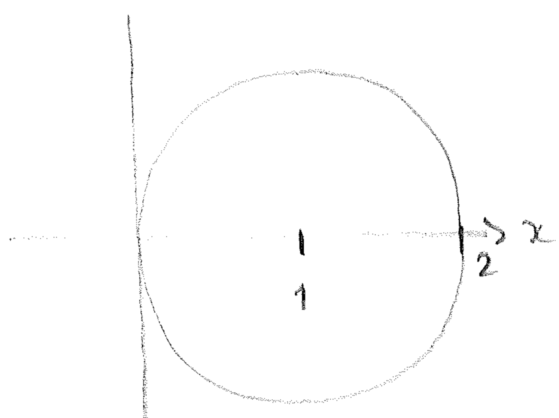
$$r\sin\theta = 1$$

$$r = \frac{1}{\sin\theta}$$



iv) $r = 2\cos\theta$

This is a circle of radius 1 centered at (1,0).



To see this, remember that $\cos\theta = \frac{x}{r}$, so

$$r = 2\cos\theta = \frac{2x}{r}$$

$$r^2 = 2x$$

$$x^2 + y^2 = 2x \quad \text{as} \quad x^2 + y^2 = r^2$$

Then

$$x^2 - 2x + y^2 = 0$$

Complete the square

$$x^2 + 2(-1)x + y^2 = 0$$

$$x^2 + 2(-1)x + (-1)^2 - (-1)^2 + y^2 = 0$$

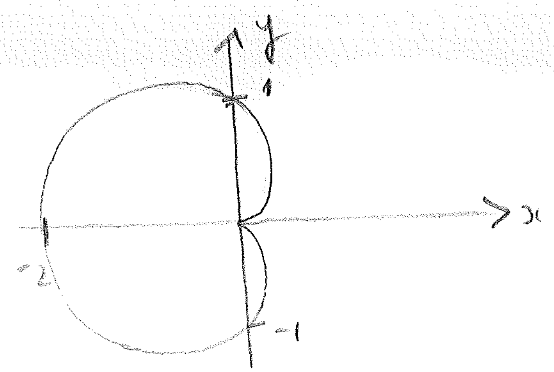
$$(x-1)^2 - 1 + y^2 = 0$$

$$(x-1)^2 + y^2 = 1$$

which is indeed the eqn. of a circle of radius 1 with centre $(1, 0)$.

v) $r = 1 - \cos\theta$

This curve is called a cardioid



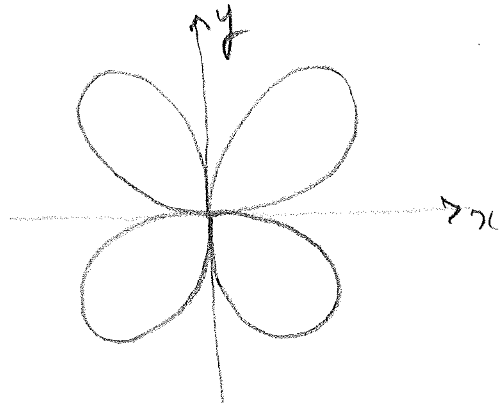
More generally, curves of the form

$$r = a + b\sin\theta \quad \text{or} \quad r = a + b\cos\theta$$

are called limaçons.

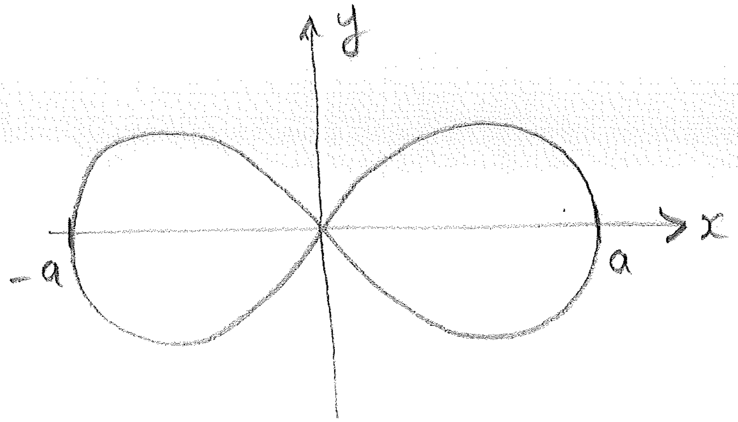
vi) $r = 3\sin 2\theta$

This curve is called a four-leaved rose



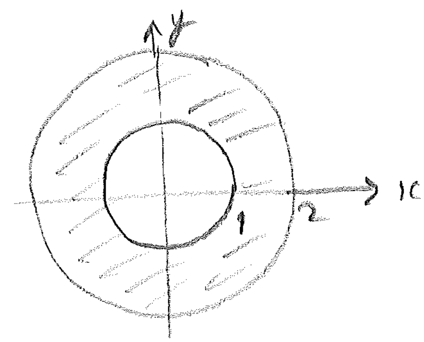
vii) $r^2 = a^2 \cos 2\theta$ ($r = 0$ if $\cos 2\theta < 0$)

This curve is called a lemniscate



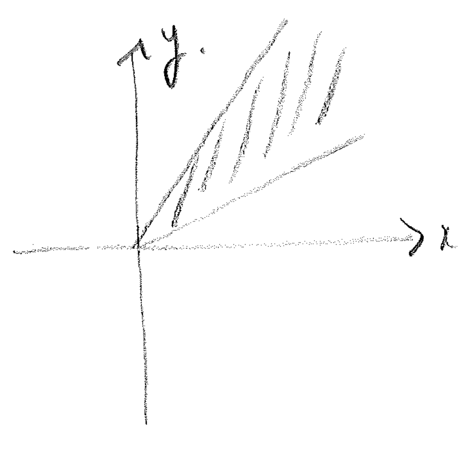
Regions in Polar Coordinates

i). $1 \le r \le 2$



Annulus (ring) of inner radius 1 and outer radius 2.

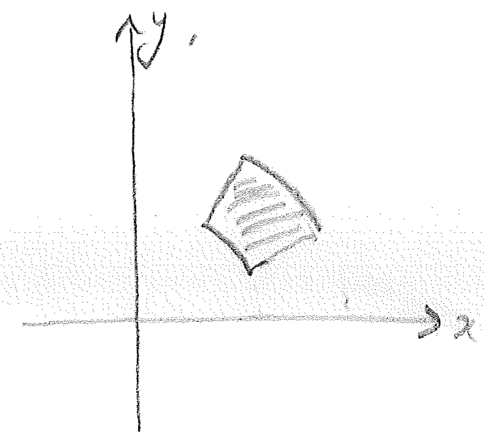
ii) $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$



Sector of angle $\frac{\pi}{6}$ (30°) between $\frac{\pi}{6}$ and $\frac{\pi}{3}$

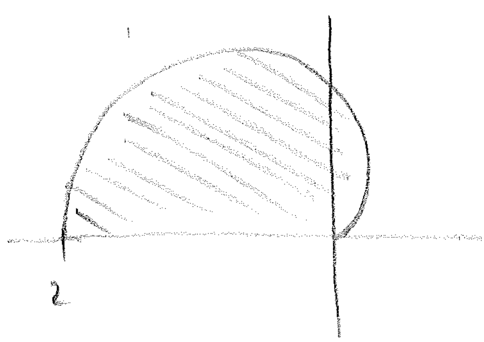
iii) $1 \leq r \leq 2, \quad \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$

Curvy rectangle



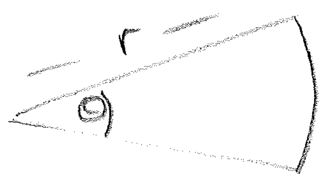
iv) $0 \leq \theta \leq \pi, \quad r < 1 - \cos\theta$

Interior of half a cardroid



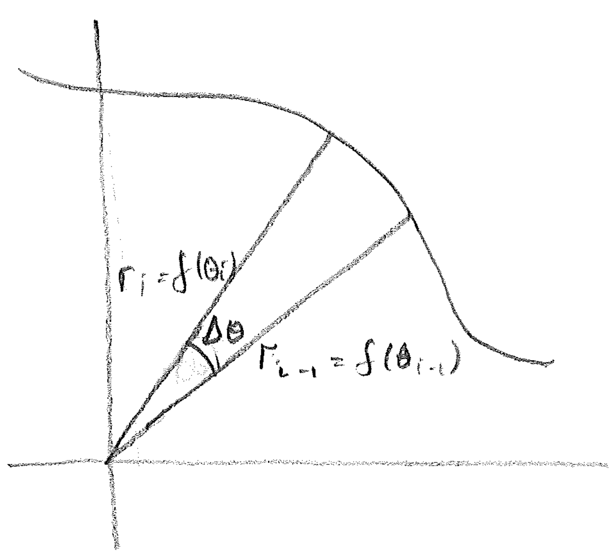
Area in Polar Coordinates

Recall that the area of a sector of radius r and angle θ is $\frac{1}{2} r^2 \theta$.



Suppose now we have a polar graph

$$r = f(\theta), \qquad \alpha \le \theta \le \beta \quad (\beta - \alpha \le 2\pi)$$



The area of the small part of the graph is approx that of a sector of radius

$$r_{i-1} \doteq f(\theta_{i-1})$$

and angle $\Delta\theta$ while the area of such a sector is

$$\frac{1}{2} r_{i-1}^2 \Delta\theta = \frac{(f(\theta_{i-1}))^2}{2} \Delta\theta.$$

Add these contributions together to get an approximation to the total area A

$$A \approx \frac{1}{2}\sum_{j=1}^{n} \left(f(\theta_{j-1})\right)^2 \Delta\theta$$

$$= \frac{1}{2}\sum_{i=0}^{n-1} \left(f(\theta_i)\right)^2 \Delta\theta$$

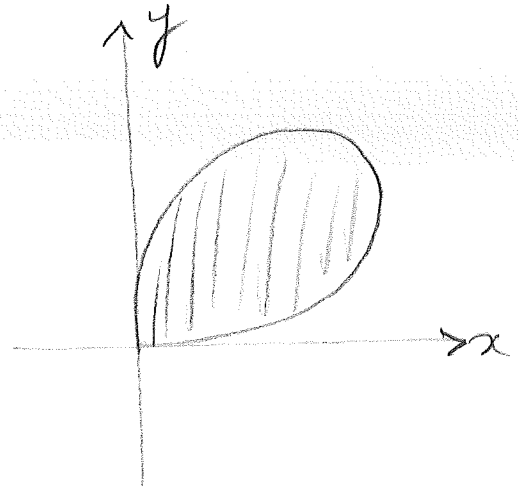
(let $i = j - 1$
as j goes from 1 to n
i goes from 0 to $n-1$)

This Riemann sum becomes an integral as we let $n \to \infty$ and we <u>define</u>

$$A := \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 \, d\theta.$$

$$= \frac{1}{2}\int_{\alpha}^{\beta} r^2(\theta) \, d\theta$$

Ex. Find the area of one petal of the four-leaved rose $r = 3\sin 2\theta$.



Here $0 \le \theta \le \frac{\pi}{2}$ and

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2(\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 9\sin^2(2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 9 \cdot \frac{1}{2}(1 - \cos(4\theta))\,d\theta$$

using $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

and $4\theta = 2(2\theta)$.

$$= \frac{9}{4} \int_0^{\frac{\pi}{2}} (1 - \cos 4\theta)\, d\theta$$

$$= \frac{9}{4} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{9}{4} \left(\left(\frac{\pi}{2} - \frac{\sin 2\pi}{4} \right) - \left(0 - \frac{\sin 0}{4} \right) \right)$$

$$= \frac{9}{4} \left(\frac{\pi}{2} - 0 - 0 \right)$$

$$= \frac{9\pi}{8}.$$

Slope in Polar Coordinates

A polar curve $r = f(\theta)$ can be expressed as a parametric Cartesian curve with parameter θ, since

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

If y is a fn of x, then by the Leibniz form of the chain rule

$$\frac{dy}{d\theta} = \frac{dy}{dx}\cdot\frac{dx}{d\theta}$$

and so if $\frac{dx}{d\theta} \neq 0$, we can divide by $\frac{dx}{d\theta}$ to get

$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{dy}{dx}, \quad \text{or} \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Ex. Find the slope of the curve

$$y = 3\sin 2\theta \quad \text{at} \quad \theta = \frac{\pi}{3}.$$

$$x = r\cos\theta = 3\sin 2\theta \cos\theta$$

so

$$\frac{dx}{d\theta} = 6\cos 2\theta \cos\theta - 3\sin 2\theta \sin\theta \quad \text{by product rule.}$$

Similarly

$$y = r\sin\theta = 3\sin 2\theta \sin\theta$$

$$\frac{dy}{d\theta} = 6\cos 2\theta \sin\theta + 3\sin 2\theta \cos\theta$$

Thus

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{6\cos 2\theta \sin\theta + 3\sin 2\theta \cos\theta}{6\cos 2\theta \cos\theta - 3\sin 2\theta \sin\theta}$$

When $\theta = \frac{\pi}{3}$ $\quad \cos\theta = \frac{1}{2}, \ \sin\theta = \frac{\sqrt{3}}{2}$

$2\theta = \frac{2\pi}{3}$ $\quad \cos 2\theta = -\frac{1}{2}, \ \sin\theta = \frac{\sqrt{3}}{2}.$

Ex. Find the slope of the curve

$r = 3\sin 2\theta$ at $\theta = \frac{\pi}{3}$.

$x = r\cos\theta = 3\sin 2\theta \cos\theta$

$y = r\sin\theta = 3\sin 2\theta \sin\theta$

Product rule

$\frac{dx}{d\theta} = 6\cos 2\theta \cos\theta - 3\sin 2\theta \sin\theta$

$$\frac{dy}{dx} = \frac{6\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + 3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{6\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) - 3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{\frac{\sqrt{3}}{2}\left(-3 + \frac{3}{2}\right)}{6\left(-\frac{1}{4}\right) - 3\left(\frac{3}{4}\right)}$$

$$= \frac{\frac{\sqrt{3}}{2}\left(-\frac{3}{2}\right)}{-\frac{6}{4} - \frac{9}{4}}$$

$$= \frac{\sqrt{3}/\cancel{2} \cdot \frac{3}{\cancel{2}}}{15/\cancel{4}}$$

$$= \frac{3\sqrt{3}}{15}$$

$$= \frac{\sqrt{3}}{5}$$

Arc Length in Polar Coordinates

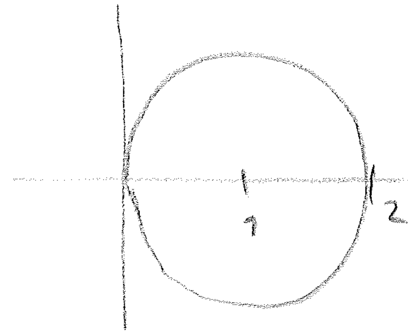
Since for $r = f(\theta)$, $\alpha \leq \theta \leq \beta$
we have the parametric form

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta,$$

the arc length is given by the usual formula for a parametric curve.

$$S = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Ex. $r = 2\cos\theta$



Here $x = 2\cos\theta\cos\theta = 2\cos^2\theta$

$y = 2\cos\theta\sin\theta$

$\frac{dx}{d\theta} = -4\cos\theta\sin\theta = -2\sin 2\theta$

as $\sin 2\theta = 2\sin\theta\cos\theta$

$\frac{dy}{d\theta} = -2\sin^2\theta + 2\cos^2\theta = 2\cos 2\theta$

as $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$= \sqrt{4\sin^2 2\theta + 4\cos^2 2\theta}$$

$$= \sqrt{4(\sin^2\theta + \cos^2\theta)}$$

$$= \sqrt{4}$$ as $\cos^2\theta + \sin^2\theta = 1$

$$= 2$$

So

$$s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} 2\, d\theta$$

$$= \left[2\theta\right]_0^{2\pi}$$

$$= 4\pi$$