§7.2 Integration by Parts

Substitution came from the chain rule.
From the product rule we get integration by parts.

Let $u = u(x)$, $v = v(x)$ be two diff fns of x.

By the product rule

$$\frac{d}{dx}(uv) = u'(x)v(x) + u(x)v'(x)$$

Rearrange.

$$-u'(x)v'(x) = -\frac{d}{dx}(uv) + u'(x)v(x)$$

$$u(x)v'(x) = \frac{d}{dx}(uv) - u'(x)v(x)$$

↑
Swap order of product

So

$$u(x)\,v'(x) = \frac{d}{dx}(uv) - v(x)\,u'(x)$$

order swapped!

Now take antids. of both sides

$$\int u(x)\,v'(x)\,dx = \int \frac{d}{dx}(uv) - \int v(x)\,u'(x)\,dx$$

Since $\int f'(x)\,dx = f(x) + C$ for any cts. $f^{\underline{n}}$ f, we have

$$\boxed{\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int v(x)\,u'(x)\,dx.}$$

Note that since the lhs & rhs involve indefinite integrals, the equality is only up to an arbitrary constant.

If we now let

$$du = u'(x)\,dx, \quad dv = v'(x)\,dx$$

(like we did in substitution), we get:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

This is the most common version of the integration by parts formula.

The idea is to choose u and v so that $\int v\,du$ is simpler than $\int u\,dv$.

This requires PRACTICE!

Ex.

$$\int x e^x \, dx$$

Let $u = x$, $\quad dv = e^x \, dx$

Then $du = dx$, $\quad v = e^x$ (no need for a constant of integration!)

and so

$$\int \underbrace{x}_{u} \underbrace{e^x \, dx}_{dv} = \underbrace{x}_{u} \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \underbrace{dx}_{du}$$

$$= x e^x - e^x + C.$$

Note how the $\int$ on the rhs, $\int e^x \, dx$ was simpler than $\int x e^x \, dx$. This is why this choice of u & v 'worked'.

Ex.

$$\int x^2 e^x \, dx$$

Here let

$$u = x^2, \qquad dv = e^x \, dx$$

$$du = 2x\,dx, \qquad v = e^x$$

So by parts

$$\int \underbrace{x^2}_{u} \underbrace{e^x \, dx}_{dv} = \underbrace{x^2}_{u} \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \cdot \underbrace{2x \, dx}_{du}$$

$$= x^2 e^x - 2 \int x e^x \, dx$$

The integral on the rhs can be done using another integration by parts ($u = x$, $dv = e^x \, dx$) or we can just use the last example to get

$$= x^2 e^x - 2 \left(x e^x - e^x \right) + C$$

$$= x^2 e^x - 2x e^x + 2e^x + C$$

This last example illustrates two important features of integration by parts.

1. It is often necessary to do more than one integration by parts to get the result (rather like l'Hôpital).

2. When one part of the integrand is a polynomial in x, it is often a good idea to let u be that polynomial and use repeated integrations by parts to differentiate away all the powers of x.

Integration by parts also works for definite integrals. The formula is

$$\boxed{\int_a^b u(x)\,v'(x)\,dx = \Big[u(x)\,v(x)\Big]_a^b - \int_a^v v(x)\,u'(x)\,dx}$$

Ex.

$$\int_2^3 \ln x \, dx$$

Doesn't look very promising. Really only one possible choice, make $u = \ln x$ as we can't integrate this (yet!)

$$u = \ln x, \quad dv = dx$$

$$du = \frac{1}{x} dx, \quad v = x.$$

So

$$\int_2^3 \underbrace{\ln x}_{u} \, \underbrace{dx}_{dv} = \Big[\underbrace{x}_{u} \, \underbrace{\ln x}_{v} \Big]_2^3 - \int_2^3 \underbrace{x}_{v} \cdot \underbrace{\frac{1}{x} dx}_{du}$$

$$= \big[x \ln x \big]_2^3 - \int_2^3 dx$$

$$= 3\ln 3 - 2\ln 2 - \big[x \big]_2^3$$

$$= 3\ln 3 - 2\ln 2 - (3-2)$$

$$= 3\ln 3 - 2\ln 2 - 1$$

This example illustrates another important rule:

Whatever you let dv be, you need to be able to find v easily.

Ex.

$$\int \cos^2\theta \, d\theta$$

Write this as

$$\int \cos\theta \cdot \cos\theta \, d\theta.$$

This suggests the choice

$$u = \cos\theta, \quad dv = \cos\theta \, d\theta$$
$$du = -\sin\theta \, d\theta, \quad v = \sin\theta.$$

Then

$$\int \cos^2\theta \, d\theta = \cos\theta \sin\theta - \int \sin\theta \cdot -\sin\theta \, d\theta$$

$$= \cos\theta \sin\theta + \int \sin^2\theta \, d\theta$$

$$= \cos\theta \sin\theta + \int (1 - \cos^2\theta) \, d\theta$$

using the identity
$\cos^2\theta + \sin^2\theta = 1$

$$= \cos\theta \sin\theta + \int d\theta - \int \cos^2\theta \, d\theta$$

Looks like we're back where we started, only not quite!

$$\int \cos^2\theta \, d\theta = \cos\theta \sin\theta + \int d\theta - \int \cos^2\theta \, d\theta.$$

Add $\int \cos^2\theta \, d\theta$ to both sides

$$2\int \cos^2\theta \, d\theta = \cos\theta \sin\theta + \int d\theta$$

So $$\int \cos^2\theta \, d\theta = \frac{\cos\theta \sin\theta}{2} + \frac{1}{2}\int d\theta$$

$$= \frac{\cos\theta \sin\theta}{2} + \frac{\theta}{2} + C.$$

n.b. this integral can be done more easily using the identity

$$\cos^2\theta = \frac{1}{2}\left(1 + \cos(2\theta)\right)$$

which comes from the double angle formulae.

Ex.

$$\int e^{2x} \sin(3x)\,dx.$$

Let $u = e^{2x}$, $dv = \sin(3x)dx$ [could also use $u = \sin 3x$, $v = e^{2x}dx$]

$du = 2e^{2x}dx$, $v = -\frac{1}{3}\cos(3x)$

$$= -\frac{1}{3}e^{2x}\cos(3x) - \int -\frac{1}{3}\cos(3x) \cdot 2e^{2x}\,dx$$

$$= -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}\int e^{2x}\cos(3x)\,dx.$$

For the $\int$ on the rhs, do another $\int$ by parts with.

$u = e^{2x}$, $dv = \cos(3x)dx$

$du = 2e^{2x}dx$, $v = \frac{1}{3}\sin(3x)$

No choice this time! See later.

$$= -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}\left\{\frac{1}{3}e^{2x}\sin(3x) - \int \frac{1}{3}\sin(3x) \cdot 2e^{2x}dx\right\}$$

Multiply out

$$= -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x) - \frac{4}{9}\int e^{2x}\sin(3x)\,dx.$$

So

$$\int e^{2x}\sin(3x)\,dx = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x) - \frac{4}{9}\int e^{2x}\sin(3x)\,dx$$

Note that the rhs. contains a multiple of the original integral. (this is good)!

Now add $\frac{4}{9}\int e^{2x}\sin(3x)\,dx$ to both sides

to get

$$\frac{13}{9}\int e^{2x}\sin(3x)\,dx = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x) + c'$$

So $$\int e^{2x}\sin(3x)\,dx = \frac{-3}{13}e^{2x}\cos(3x) + \frac{2}{13}e^{2x}\sin(3x) + C,$$

$$C = \frac{9c'}{13}.$$

Important Note for Integrals of the Type

$$\int e^{ax} \sin bx \, dx, \quad \int e^{ax} \cos bx \, dx$$

You need two integrations by parts and in the first integration by parts you can let u be either the exponential fn or the trig fn.

However the <u>second</u> integration by parts must be 'in the same direction' as the first. In other words if u is the exp. fn in the first integration, then the new u for the second should again be the exp. fn. Similarly if you pick the trig fn for u in the first integration, you need to pick the trig fn again for u in the second.

If you don't do this, you end up undoing the first integration and saying something like

$$\int e^{ax} \sin bx \, dx = \int e^{ax} \sin bx \, dx$$

which is true, but useless. Try it!

After the second integration by parts, a constant multiple of the original integral appears on the rhs.

This gives an <u>equation</u> in the desired integral which can then be solved by <u>simple</u> algebra.