§ 11.4 Separation of Variables

Recall that in § 11.2 we looked the DE

$$\frac{dy}{dx} = -\frac{x}{y}$$

and guessed that the sol^ns were circles of the form

$$x^2 + y^2 = C.$$

One can check that these are sol^ns by differentiation:

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(C)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

However, there is also a way to get the solns from the original eqn.

$$\frac{dy}{dx} = -\frac{x}{y}$$

via an algebraic trick.

<u>Trick</u>. Treat $\frac{dy}{dx}$ like a fraction and rearrange the eqn. so that everything involving y is on the left and everything involving x is on the right.

$$dy = -\frac{x}{y}\,dx$$

$$y\,dy = -x\,dx$$

This trick is known as the <u>separation of variables</u> (for obvious reasons!).

We now integrate both sides of the rearranged eq^{n}

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + k, \qquad k \text{ constant}$$

$$y^2 = -x^2 + C \qquad (C = 2k)$$

$$x^2 + y^2 = C.$$

N.b. this trick doesn't always work – one can't always separate x and y. Neverthess, it is a very useful short cut when it does work.

Exponential Growth and Decay

Consider the DE

$$\frac{dy}{dx} = ky$$

Separating the variables gives

$$\frac{dy}{y} = k\,dx$$

We then integrate both sides

$$\int \frac{dy}{y} = \int k\,dx$$

$$\ln|y| = kx + C, \quad C \text{ constant.}$$

Taking the exponential of both sides

$$e^{\ln|y|} = e^{kx + C}$$

$$|y| = e^{kx+c}$$

$$|y| = e^{kx} \cdot e^{c}.$$

So $$y = \pm e^{c} e^{kx}$$

and if we let A be the constant $\pm e^{c}$, we get

$$y = A e^{kx}.$$

If in addition, we want our solution to have a particular value y_0 at $x=0$, we have, on setting $x=0$,

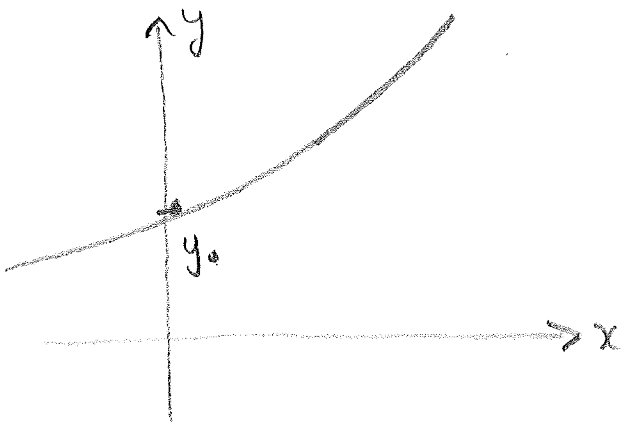
$$y_0 = A e^{k \cdot 0}$$

$$y_0 = A \cdot 1$$

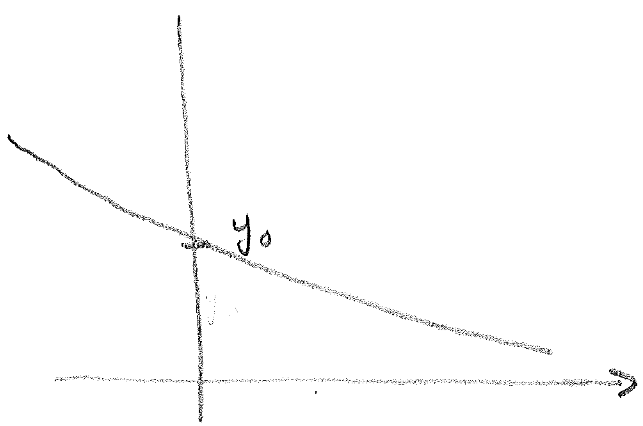
Thus $A = y_0$ and the soln of the IVP is

$$y = y_0 e^{kx}.$$

If $k > 0$, the sol^n grows without bound as $x \to \infty$ and we have <u>exponential growth</u>



If $k < 0$, the sol^n tends to 0 as $x \to \infty$ and we have <u>exponential decay</u>



Finally, if $k = 0$, then our sol^n is just the constant fn

$$y = y_0.$$

Ex. For $k > 0$, find sol^ns of

$$\frac{dH}{dt} = -k(H-20)$$

Separating variables gives

$$\frac{dH}{H-20} = -k\,dt$$

and if we then integrate both sides, we get

$$\int \frac{dH}{H-20} = \int -k\,dt$$

$$\ln|H-20| = -kt + C \quad ; \quad C \text{ constant}$$

$$e^{\ln|H-20|} = e^{-kt+C}$$

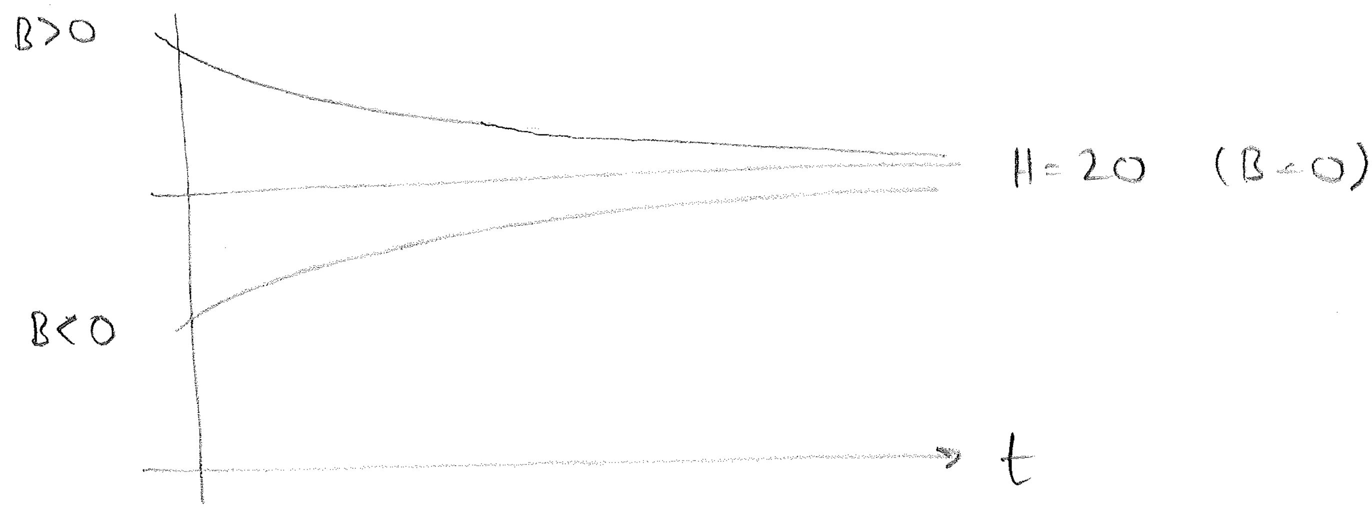
$$|H-20| = e^{C} \cdot e^{-kt}$$

$$H-20 = \pm e^{C} \cdot e^{-kt}$$

$$H - 20 = B e^{-kt} \quad \text{where} \quad B = \pm e^{c}$$

$$H = 20 + B e^{-kt}$$

Solns look like



An eqn of this type could be used to model a situation where an object reaches thermal equilibrium (e.g. a beer left out in the garden when the temp. is 20°C).

Ex. $\frac{dP}{dt} = 2P - 2Pt$, $P(0) = 5$.

$$= P(2-2t)$$

Separating variables gives

$$\frac{dP}{P} = (2-2t)\,dt$$

Integrate.

$$\int \frac{dP}{P} = \int (2-2t)\,dt$$

$$\ln|P| = 2t - t^2 + C , \quad C \text{ constant.}$$

$$|P| = e^{2t-t^2+C}$$

$$P = \pm e^{C} \cdot e^{2t-t^2}$$

$$P = Ae^{2t-t^2}$$

Now use the IC $P(0) = 5$

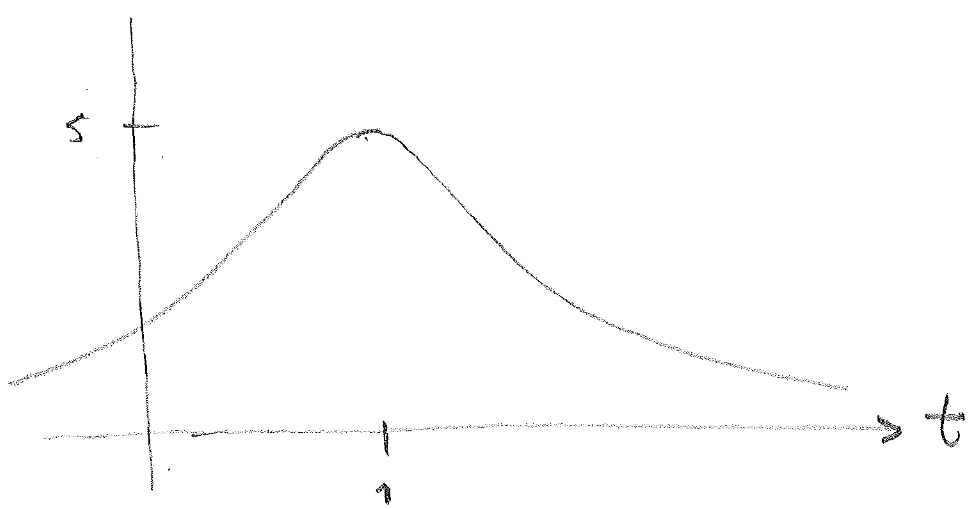
$$5 = A e^{2(0) - 0^2}$$

$$5 = A$$

So $$P = 5 e^{2t - t^2}$$

$$= 5 e^{1 - (1-t)^2}$$

Sol^n is a bell-shaped curve (similar to the normal distribution in statistics) with the peak at $t = 1$.



Ex $\frac{dy}{dt} = y^2(1+t)$, $y(1) = 2$.

Separate the variables

$$\frac{dy}{y^2} = (1+t)\,dt$$

Integrate

$$\int \frac{dy}{y^2} = \int (1+t)\,dt$$

$$-\frac{1}{y} = t + \frac{t^2}{2} + C$$

$$y = \frac{-1}{t + t^2/2 + C}$$

Apply the IC to find C

$$y(1) = 2 \Rightarrow 2 = \frac{-1}{1 + \frac{1}{2} + C}$$

$$-\frac{1}{2} = \frac{3}{2} + C$$

$$C = -2.$$

The solⁿ to the IVP is then

$$y = \frac{-1}{t + t^2/2 - 2}$$

$$= \frac{1}{2 - t - t^2/2}$$

$$= \frac{2}{4 - 2t - t^2}.$$

Justification for Separation of Variables

Suppose we can write our DE in the form

$$\frac{dy}{dx} = g(x)\,f(y)$$

Provided $f(y) \neq 0$, we can define $h(y) = \frac{1}{f(y)}$ and rewrite the eq^n as

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

Now multiply both sides by $h(y)$

$$h(y)\frac{dy}{dx} = g(x).$$

Next integrate both sides wrt x.

$$\int h(y)\frac{dy}{dx}\,dx = \int g(x)\,dx.$$

If we regard y as a fn of x, then for the left hand integral, we can treat y as a substitution since

$$dy = y'(x)\,dx = \frac{dy}{dx} \circ dx .$$

Using this we can then rewrite the eqn as

$$\int h(y)\,dy = \int g(x)\,dx$$

which is the same as we would have got (much more quickly!) from separation of variables.