§10.4 The Error in Taylor Polynomial Approximations

Suppose we want to approximate the values of some fn $f(x)$ near $x=a$ using the n-th degree Taylor polynomial $P_n(x)$. In order to know whether we have a good approximation we need to look at the error

$$E_n(x) = f(x) - P_n(x).$$

Of course we want the size of $E_n(x)$ to be small and we do this by finding an upper bound for $|E_n|$. The idea then is that if this upper bound is small, then E_n has to also be small.

Finding an Error Bound

For the sake of simplicity, assume for now that $a = 0$ (the case $a \neq 0$ is very similar)

Recall that we constructed $P_n(x)$ so that the first n derivatives of P_n at $x = 0$ were the same as those of f.

Thus

$$E_n(0) = f(0) - P_n(0) = 0$$

$$E_n'(0) = f'(0) - P_n'(0) = 0$$

$$E_n''(0) = f''(0) - P_n''(0) = 0$$

$$\vdots$$

$$E_n^{(n)}(0) = f^{(n)}(0) - P^{(n)}(0) = 0$$

If we now take one more derivative, then, since $P_n(x)$ is a polynomial of degree n, $P_n^{(n+1)}(x) = 0$, and so

$$E_n^{(n+1)}(x) = f^{(n+1)}(x).$$

Suppose now that we know $|f^{(n+1)}(x)| \leq M$ for some constant M and for some range of non-negative values of x, say for $0 \leq x \leq \delta$. (case for negative values is similar)

Thus

$$-M \leq f^{(n+1)}(x) \leq M \quad \text{for} \quad 0 \leq x \leq \delta.$$

Since $E_n^{(n+1)}(x) = f^{(n+1)}(x)$, we have

$$-M \leq E_n^{(n+1)}(x) \leq M \quad \text{for} \quad 0 \leq x \leq \delta.$$

The idea is then to go from a bound on the $(n+1)$st derivative $E_n^{(n+1)}(x)$ to a bound on $E_n(x)$ by repeated integrations from 0 to x.

Thus

$$-\int_0^x M\,dt \leq \int_0^x E_n^{(n+1)}(t)\,dt \leq \int_0^x M\,dt, \quad 0 \leq x \leq \delta$$

By the First Fundamental Theorem of Calculus, since $E_n^{(n)}(x)$ is an antid. of $E_n^{(n+1)}(x)$, this gives

$$\left[-Mt\right]_0^x \leq \left[E_n^{(n)}(t)\right]_0^x \leq \left[Mt\right]_0^x$$

$$-Mx - 0 \leq E_n^{(n)}(x) - E_n^{(n)}(0) \leq Mx - 0$$

and since $E_n^{(n)}(0) = f^{(n)}(0) - P_n^{(n)}(0) = 0$, we have

$$-Mx \leq E_n^{(n)}(x) \leq Mx, \qquad 0 \leq x \leq \delta$$

Now integrate again from 0 to x

$$\int_0^x -Mt\,dt \leq \int_0^x E_n^{(n)}(t)\,dt \leq \int_0^x Mt\,dt.$$

and by FToC I again

$$\left[-\frac{Mt^2}{2}\right]_0^x \leq \left[E_n^{(n-1)}(t)\right]_0^x \leq \left[\frac{Mt^2}{2}\right]_0^x$$

and since $E_n^{(n-1)}(0) = f^{(n-1)}(0) - P_n^{(n-1)}(0) = 0$,

$$-\frac{Mx^2}{2} \leq E_n^{(n-1)}(x) \leq \frac{Mx^2}{2}, \quad 0 \leq x \leq \delta$$

If we keep going like this, we eventually undo all the derivatives and in the end we get

$$-\frac{1}{(n+1)!} Mx^{n+1} \leq E_n(x) \leq \frac{1}{(n+1)!} Mx^{n+1}, \quad 0 \leq x \leq \delta.$$

Thus

$$|E_n(x)| = |f(x) - P_n(x)| \leq \frac{1}{(n+1)!} Mx^{n+1}, \quad 0 \leq x \leq \delta.$$

We then get.

The Lagrange Error Bound for $P_n(x)$

Suppose f and all its derivatives are cts.
If $P_n(x)$ is the nth Taylor polynomial
for $f(x)$ about $x = a$, then

$$|E_n(x)| = |f(x) - P_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

where $|f^{(n+1)}(t)| \leq M$ on the interval between
a and x.

Using the Lagrange Error Bound

Ex. Give a bound on E_4 when e^x is approximated by $P_4(x)$ about $x=0$ on the interval $-.5 \le x \le .5$.

Let $f(x) = e^x$.

Then $f^{(5)}(x) = e^x$ and since e^x is positive and increasing

$$|f^{(5)}(x)| \le e^{0.5} = \sqrt{e} < 2, \quad -.5 \le x \le .5.$$

Thus we can take $M = 2$ which gives

$$|E_4| = |e^x - P_4(x)| \le \frac{2}{5!}|x|^5, \quad -.5 \le x \le .5.$$

Thus, for $-.5 \le x \le .5$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

and the error is at most $\frac{2}{120}(.5)^5 < .0006$.

Of course, for smaller values of x, we get smaller errors.

e.g $\quad E(.2) \approx 2.75 \times 10^{-6}$

while $\quad E(.1) \approx 8.47 \times 10^{-8}$

One can check that

$$E(.2)/32 \approx 8.62 \times 10^{-8} \approx E(.1).$$

This is very much what we'd expect given our bound

$$|E_4(x)| \leq \frac{1}{60}|x|^5$$

since decreasing the value of x by a factor of 2 will decrease the value of the bound by a factor of $2^5 = 32$.

Ex. Show that the Taylor series for $\cos x$ converges to $\cos x$ everywhere on $\mathbb{R}$.

Letting $f(x) = \cos x$

$$f^{(n)}(x) = \begin{cases} \cos x \\ -\sin x \\ -\cos x \\ \sin x \end{cases}, \quad \text{depending on the value of } n.$$

Thus $|f^{(n+1)}(x)| \leq 1$ on $\mathbb{R}$ and using the Lagrange error bound (on any interval about 0 containing x), with $M = 1$,

$$|E_n(x)| = |\cos x - P_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

One can show that for any fixed $a > 0$

$$\lim_{n \to \infty} \frac{a^n}{n!} = 0$$

and hence $|E_n(x)| \longrightarrow 0$ as $n \to \infty$.

Since x was an arbitrary real number, $P_n(x) \longrightarrow \cos x$ as $n \to \infty$ everywhere on $\mathbb{R}$ as required.