§10.2 Taylor Series

Taylor series are the power series we obtain from some function $f(x)$ near some pt. $x=a$ when we let the degree n of the Taylor polynomials P_n tend to infinity.

For example, we saw in the last section for $f(x) = e^x$ near $x=0$, we had.

$$e^x \approx P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

It turns out that the larger we make n, the better $P_n(x)$ is as an approximation to e^x. In fact, in the limit as $n \to \infty$, $P_n(x)$ converges to e^x and we have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The same thing happens with the Taylor polynomials for $\sin x$ and $\cos x$ and we get

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

<u>Note:</u> These Taylor series converge to their respective f^{n}s for every value of x.

Taylor Series in General

Suppose $f(x)$ is infinitely differentiable at $x=a$ (i.e. $f^{(n)}(a)$ exists for every $n \geq 0$).

In this case, we can form the Taylor series in a similar way to the Taylor polynomials. However, the Taylor series may not converge to $f(x)$ for every value of x.

If the series does converge to $f(x)$, then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

When $a=0$, we get

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

which is sometimes called the McLaurin series for f.

As we noted before, the Taylor series for f is a power series and the partial sums for this power series are the Taylor polynomials.

The two related questions we need to ask are

1. Where does the Taylor series converge?

2. Where does the Taylor series converge to $f(x)$?

Since power series converge on an interval (as we saw in § 9.5), we would expect an interval of convergence here as well.

Ex. $\ln x$ about $x = 1$.

We showed in the last section that the Taylor polynomials are given by

$$\ln x \approx P_n(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + (-1)^{n-1}\frac{(x-1)^n}{n}$$

A similar calculation gives the Taylor series

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + (-1)^{n-1}\frac{(x-1)^n}{n} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}.$$

If we then let $a_n = \frac{(-1)^{n-1}(x-1)^n}{n}$, then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{(-1)^n (x-1)^{n+1}}{n+1} \Big/ \frac{(-1)^{n-1}(x-1)^n}{n} \right|$$

$$= \lim_{n\to\infty} \left| (x-1) \cdot \frac{n}{n+1} \right|.$$

$$= |x-1| \lim_{n\to\infty} \left|\frac{n}{n+1}\right|$$

$$= |x-1| \lim_{n\to\infty} \left|\frac{1}{1+\frac{1}{n}}\right|$$

$$= |x-1|$$

By the ratio test, the Taylor series will then converge provided

$$|x-1| < 1$$

ie $\quad 0 < x < 2.$

When $x=0$, we have the series

$$-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\cdots = \sum_{n=1}^{\infty} -\frac{1}{n}$$

which diverges as it is a negative harmonic series.

When $x=2$, we have the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+- \quad = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

which converges by the alternating series test.

Thus the series converges for $0 < x \leq 2$ (i.e. on the interval $(0, 2]$) and diverges everywhere else.

In fact, on $(0, 2]$, the Taylor series not only converges, but it also converges to $\ln x$.

The Binomial Series Expansion.

We find the Taylor series for $f(x) = (1+x)^p$ about $x = 0$. Here p is a constant, but not necessarily a positive integer. Here

$f(x) = (1+x)^p$ so $f(0) = 1$, $C_0 = 1$

$f'(x) = p(1+x)^{p-1}$ $f'(0) = p$, $C_1 = p$

$f''(x) = p(p-1)(1+x)^{p-2}$ $f''(0) = p(p-1)$, $C_2 = \dfrac{p(p-1)}{2!}$

$f'''(x) = p(p-1)(p-2)(1+x)^{p-3}$ $f'''(0) = p(p-1)(p-2)$, $C_3 = \dfrac{p(p-1)(p-2)}{3!}$

etc.

Thus the Taylor series for f is

$$1 + px + \frac{p(p-1)x^2}{2!} + \frac{p(p-1)(p-2)x^3}{3!} + \cdots$$

$$\cdots + \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}x^n + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!}x^n$$

If we let $a_n = \dfrac{p(p-1)\cdots(p-n+1)}{n!}x^n$, then

$$a_{n+1} = \frac{p(p-1)\cdots(p-n+1)(p-(n+1)+1)}{(n+1)!}x^{n+1}$$

$$= \frac{p(p-1)\cdots(p-n+1)(p-n)}{(n+1)!}$$

So

$$\frac{|a_{n+1}|}{|a_n|} = \left| \frac{\dfrac{p(p-1)\cdots(p-n+1)(p-n)\,x^{n+1}}{(n+1)!}}{\dfrac{p(p-1)\cdots(p-n+1)\,x^n}{n!}} \right|$$

$$= \left| \frac{p(p-1)\cdots(p-n+1)(p-n)\cdot n!}{p(p-1)\cdots(p-n+1)\cdot(n+1)!} \frac{x^{n+1}}{x^n} \right|$$

$$= \left| \frac{\cancel{p(p-1)\cdots(p-n+1)}(p-n)\cdot \cancel{n(n-1)\cdots 2\cdot 1}\cdot x^{n+1}}{\cancel{p(p-1)\cdots(p-n+1)}(n+1)\cancel{n(n-1)\cdots 2\cdot 1}\cdot x^n} \right|$$

$$= \left| \frac{p-n}{n+1} x \right|$$

$$= \left| \frac{n-p}{n+1} x \right|$$

$$= \left| \frac{1-p/n}{1+\frac{1}{n}} \right| |x| \longrightarrow |x| \quad \text{as} \quad n \to \infty.$$

By the ratio test, we then see that our binomial series converges (absolutely) if $|x|<1$ and diverges if $|x|>1$.

Again, if $|x|<1$ the series not only converges, but it converges to $(1+x)^p$ and so we can say that

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

$$+ \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + \cdots, \quad |x|<1.$$

$$= 1 + \sum_{n=1}^{\infty} \frac{p\cdots(p-n+1)}{n!}x^n, \quad |x|<1.$$

What happens at the endpts. $x=\pm 1$ needs to be checked on an individual basis.

Note that if p is a natural number, then for $n > p$

$$p - n + 1 \leqslant 0$$

and so the Taylor coefficient

$$\frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$$

must contain 0 as one of the factors in the numerator and is $\therefore$ 0.

Thus, in this case the series terminates after the term in x^p and we get the familiar expansion for $(1+x)^p$ from the binomial theorem.

Ex. Use the binomial series with $p=4$ to expand $(1+x)^4$.

$$(1+x^4) = 1 + 4x + \frac{4 \cdot 3}{2!} x^2 + \frac{4 \cdot 3 \cdot 2}{3!} x^3 + \frac{4 \cdot 3 \cdot 2 \cdot 1}{4!} x^4.$$

All other terms have 0 as a factor and are $\therefore$ 0.

Thus

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

which is the same as we'd get from the binomial theorem.

Ex. Find the Taylor series about $x=0$ for $\frac{1}{1+x}$.

$$\frac{1}{1+x} = (1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3$$

$$+ \frac{(-1)(-2)(-3)(-4)}{4!}x^4 + \cdots$$

$$= 1 - x + x^2 - x^3 + x^4 - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n, \qquad -1 < x < 1.$$