Chapter 10

Approximating Functions Using Series

§10.1 Taylor Polynomials

The idea in this section is to find a way of choosing a polynomial which gives a good approximation of the values of some function $f(x)$ near a chosen point $x = a$.

We have already seen something of this in Math 141.

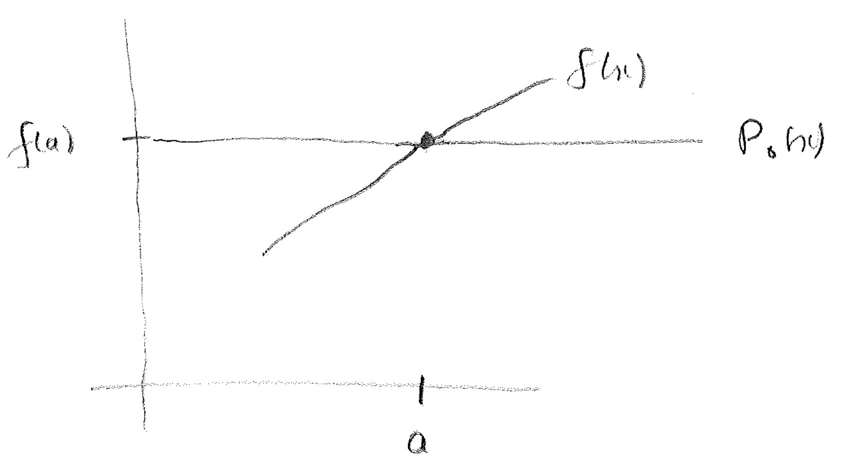
For example, if $f(x)$ is continuous (cts.) at $x = a$, then for x near a,

$$f(x) \approx f(a) \qquad \text{(ie } f(x) \text{ is close to } f(a)\text{)}$$

and so if we let $P_0(x)$ be the <u>constant</u> fn

$$P_0(x) = f(a)$$

then $P_0(x)$ gives us an approx. of f near $x = a$.



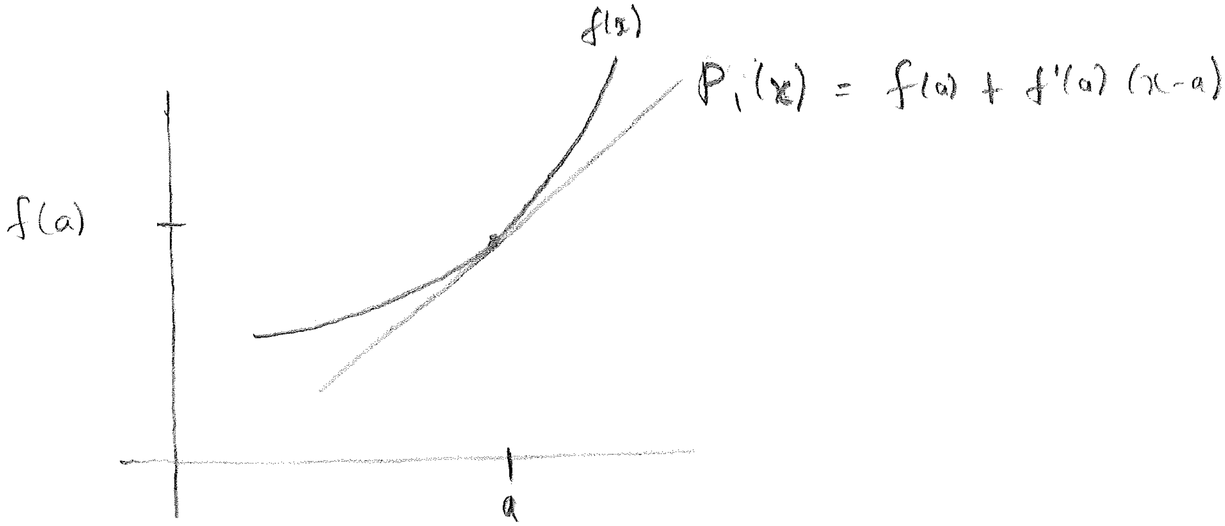
Note that $P_0(x)$ has the same value as $f(x)$ for $x = a$.

A better way to do the approximation is using linear fns.

This gives us the tangent line approximation also called the linear approximation.

Recall that if $f(x)$ is differentiable at $x = a$, then for x near a

$$f(x) \approx f(a) + f'(a)(x-a).$$



If we then let $P_1(x) = f(a) + f'(a)(x-a)$, then P_1 gives us an approx for f near $x = a$.

Note that $P_1(x)$ has the same value as $f(x)$ at $x = a$ and also the same slope of f at $x = a$.

Ex. Find the linear approx. for

$f(x) = \sin x$ near $x = 0$.

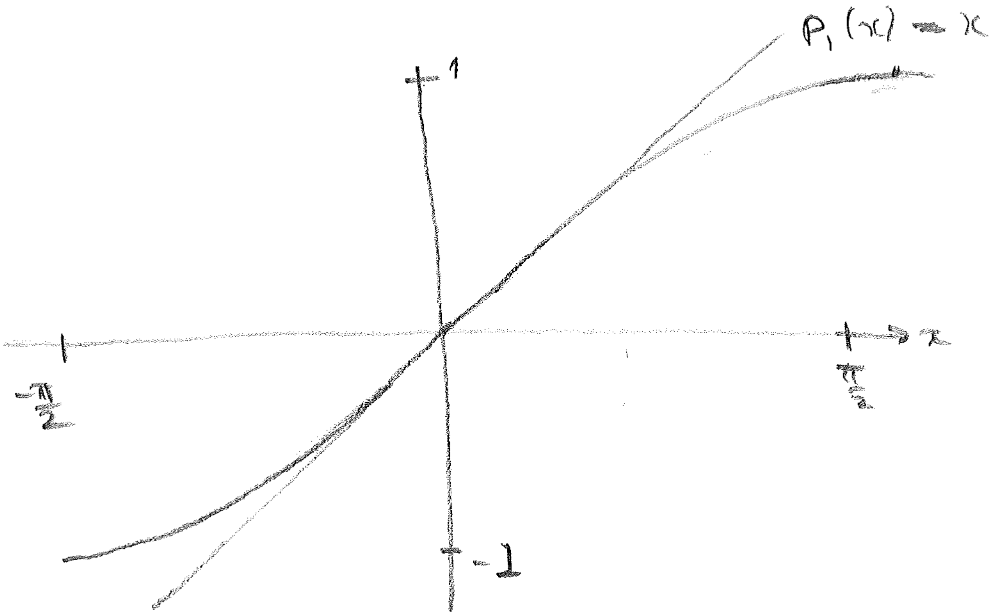
Here $f(x) = \sin x$, $f(0) = \sin(0) = 0$

$f'(x) = \cos x$, $f'(0) = \cos(0) = 1$

and so $P_1(x) = 0 + x = x$

and

$$\sin x \approx x, \quad x \text{ small}$$



e.g. $P_1(.2) = .2$ which is close to

$\sin(.2) = .1986\ldots\ldots$

Quadratic Approximations

Since a linear approx. is clearly better (more accurate) than a constant approx., it is natural to guess that we could do even better if we approximated $f(x)$ using quadratic polynomials

We choose our quadratic polynomial by extending the reasoning for the constant and linear approximations, namely we require that our quadratic have the same value, the same first derivative and the same second derivative as f does at $x = a$.

So suppose that f is twice differentiable at $x = a$ and let $P_2(x)$ be our quadratic approximation.

A useful trick here is instead of writing something like

$$P_2(x) = c_0 + c_1 x + c_2 x^2$$

(which is probably the first thing one might try), we instead take into account that we want an approximation near $x = a$ and write

$$P_2(x) = c_0 + c_1 (x-a) + c_2 (x-a)^2.$$

Note that we already did something like this for the linear approximation.

Then

$$P_2(x) = C_0 + C_1(x-a) + C_2(x-a)^2,$$

$$P_2(a) = C_0$$

and so $P_2(a) = f(a) \Rightarrow \underline{C_0 = f(a)}$

$$P_2'(x) = C_1 + 2C_2(x-a),$$

$$P_2'(a) = C_1$$

and so $P_2'(a) = f'(a) \Rightarrow C_1 = f'(a).$

$$P_2''(x) = 2C_2$$

$$P_2''(a) = 2C_2$$

and so $P_2''(a) = f''(a) \Rightarrow 2C_2 = f''(a), \; C_2 = \frac{f''(a)}{2}.$

Get

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

and so for x near a,

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Note how the constant and linear terms are the same as for the constant and linear approximations. In this sense we can say the quadratic approximation 'contains' the constant and linear approximations.

Ex. Find the quadratic approx. for $f(x) = \cos x$ near $x = 0$.

Here $f(x) = \cos x$, $f(0) = \cos(0) = 1$

and so $c_0 = f(0) = 1$.

$f'(x) = -\sin x$, $f'(0) = -\sin(0) = 0$

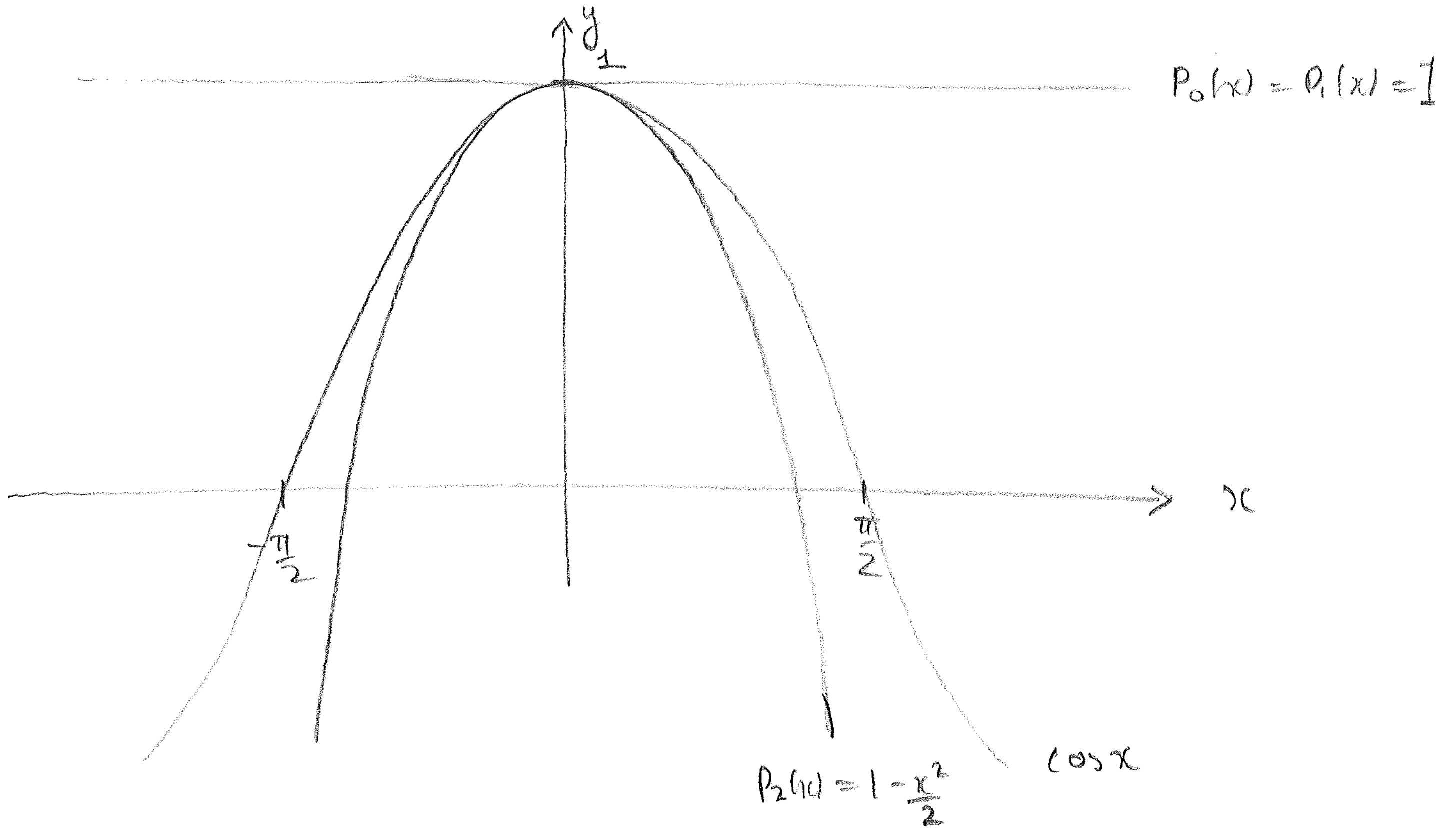
and so $c_1 = f'(0) = 0$

$f''(x) = -\cos x$, $f''(0) = -\cos(0) = -1$

and so $c_2 = \frac{f''(0)}{2} = -\frac{1}{2}$.

Thus $P_2(x) = 1 - \frac{x^2}{2}$ and

$$\cos x \approx 1 - \frac{x^2}{2}, \quad x \text{ small}$$



e.g. $P_2(.4) = .92$

which is much closer to the actual value of $\cos(.4) = .921\ldots$ than

$P_0(.4) = P_1(.4) = 1$.

Approximation Using Higher Degree Polynomials

Suppose $f(x)$ is differentiable n times at $x = a$ and we try to approxmate $f(x)$ near a with a polynomial of degree n. Similarly to before we write this polynomial as

$$P_n(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \cdots + C_n(x-a)^n.$$

As before we want $P_n(a) = f(a)$, and so setting $x = a$, we get

$$P_n(a) = C_0 + 0 + 0 + \qquad + 0 = f(a)$$

So $\quad C_0 = f(a). \qquad$ (same as before).

Now differentiate w.r.t. x to get

$$P_n'(x) = 0 + C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \qquad + nC_n(x-a)^{n-1}$$

Again, we want $P_n'(a) = f'(a)$ and so

$$P_n'(a) = C_1 + 0 + \quad + 0 = f'(a).$$

Thus $C_1 = f'(a)$ (again, same as before).

Differentiating again, we get:

$$P_n''(x) = 2C_2 + 3\cdot 2\, C_3(x-a) + \cdots + n(n-1)\,C_n(x-a)^{n-2}$$

We want $P_n''(a) = f''(a)$ and so

$$P_n''(a) = 2C_2 + 0 + \quad + 0 = f''(a).$$

Thus $2C_2 = f''(a)$

ie $C_2 = \dfrac{f''(a)}{2\cdot 1} = \dfrac{f''(a)}{2!}$ (again).

Now differentiate again to get

$$P_n'''(x) = 3 \cdot 2\, C_3 + \cdots + n(n-1)(n-2)\, C_n (x-a)^{n-3}$$

Similarly to above, we want $P_n'''(a) = f'''(a)$ and so

$$P_n'''(a) = 3 \cdot 2\, C_3 + 0 + \qquad + 0 = f'''(a).$$

Thus $\quad 3 \cdot 2\, C_3 = f'''(a)$

or

$$C_3 = \frac{f'''(a)}{3 \cdot 2 \cdot 1} = \frac{f'''(a)}{3!}$$

Continuing in this way, we would find that

$$C_4 = \frac{f^{(4)}(a)}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{f^{(4)}(a)}{4!}$$

and so on up to

$$C_n = \frac{f^{(n)}(a)}{n(n-1)(n-2)\cdots 2 \cdot 1} = \frac{f^{(n)}(a)}{n!}$$

To summarize.

If f is n times differentiable at $x=a$, then for x near a

$$f(x) \approx P_n(x)$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$P_n(x)$ is called the <u>Taylor polynomial of degree n</u> about $x=a$.

Since $0! = 1$, $1! = 1$, we can rewrite $P_n(x)$ as

$$P_n(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

Ex. Find $P_7(x)$ for $f(x) = \sin x$ about $x = 0$.

Compare $P_7(\frac{\pi}{3})$ with $\sin(\frac{\pi}{3})$.

Have

$f(x) = \sin x$, gives $f(0) = 0$, $C_0 = \frac{0}{0!} = 0$

$f'(x) = \cos x$, $f'(0) = 1$, $C_1 = \frac{1}{1!} = 1$

$f''(x) = -\sin x$, $f''(0) = 0$, $C_2 = \frac{0}{2!} = 0$

$f'''(x) = -\cos x$, $f'''(0) = -1$, $C_3 = \frac{-1}{3!}$

$f^{(4)}(x) = \sin x$, $f^{(4)}(0) = 0$, $C_4 = \frac{0}{4!} = 0$

$f^{(5)}(x) = \cos x$, $f^{(5)}(0) = 1$, $C_5 = \frac{1}{5!}$

$f^{(6)}(x) = -\sin x$ $f^{(6)}(0) = 0$, $C_6 = \frac{0}{6!} = 0$

$f^{(7)}(x) = -\cos x$ $f^{(7)}(0) = -1$, $C_7 = \frac{-1}{7!}$

Thus, for x near 0

$$\sin x \approx P_7(x) = 0 + 1 \cdot x + 0x^2 - \frac{1}{3!}x^3 + 0x^4 + \frac{1}{5!}x^5 + 0x^6 - \frac{1}{7!}x^7$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

At $\frac{\pi}{3} = 1.0471976\ldots$

$$P_7\left(\frac{\pi}{3}\right) = 0.8660213\ldots$$

while

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} = 0.8660254\ldots$$

Ex. Find $P_n(x)$ for $f(x) = e^x$ about $x = 0$ (any n).

Since $\frac{d}{dx}(f(x)) = \frac{d}{dx}(e^x) = e^x = f(x)$,

for any i

$$f^{(i)}(0) = e^0 = 1.$$

Thus $$C_i = \frac{f^{(i)}(0)}{i!} = \frac{1}{i!}$$

and so

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

and for x small

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

Ex Find $P_n(x)$ for $f(x) = \frac{1}{1-x}$, x near 0.

$f(x) = \frac{1}{1-x}$, $f(0) = 1$, so $c_0 = 1$

$f'(x) = \frac{-1}{(1-x)^2}(-1)$

$= \frac{1}{(1-x)^2}$, $f'(0) = 1$, so $c_1 = 1$.

$f''(x) = \frac{-2}{(1-x)^3} \cdot (-1)$

$= \frac{2}{(1-x)^3}$, $f''(0) = 2$, so $c_2 = \frac{2}{2} = 1$.

$f'''(x) = \frac{-6}{(1-x)^4}(-1)$

$= \frac{6}{(1-x)^4}$, $f'''(0) = 6$, so $c_3 = \frac{6}{3!} = 1$.

Continuing in this way, we get $f^{(4)}(0) = 24 = 4!$, $f^{(5)}(0) = 120 = 5!$ and so on up to

$f^{(n)}(0) = n!$

Thus

$$c_i = \frac{f^{(i)}(0)}{i!} = \frac{i!}{i!} = 1$$

and so

$$P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$$

and for x small

$$\frac{1}{1-x} \approx 1 + x + x^2 + \cdots + x^n.$$

Note that this is pretty much what we'd expect given that for $-1 < x < 1$, we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

<u>Ex</u> Find $P_n(x)$ for $f(x) = \ln x$ for x near 1.

$f(x) = \ln x$, so $f(1) = \ln 1 = 0 \Rightarrow c_0 = 0$

$f'(x) = \frac{1}{x}$, $f'(1) = \frac{1}{1} = 1 \Rightarrow c_1 = 1$.

$f''(x) = -\frac{1}{x^2}$, $f''(1) = -1 \Rightarrow c_2 = \frac{-1}{2!} = \frac{-1}{2}$

$f'''(x) = \frac{2}{x^3}$, $f'''(1) = 2 \Rightarrow c_3 = \frac{2}{3!} = \frac{1}{3}$

$f^{(4)}(x) = \frac{-6}{x^4}$, $f^{(4)}(1) = -6 \Rightarrow c_4 = \frac{-6}{4!} = \frac{-1}{4}$

Continuing in this way, we see that

$f^{(i)}(1) = (-1)^{i-1} \cdot (i-1)!$, so

$$c_i = \frac{f^{(i)}(1)}{i!} = \frac{(-1)^{i-1}(i-1)!}{i!} = \frac{(-1)^{i-1}(i-1)(i-2)\cdots 2 \cdot 1}{i(i-1)(i-2)\cdots 2 \cdot 1}$$

$$= \frac{(-1)^{i-1}}{i}.$$

Thus

$$P_n(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + \frac{(-1)^{n-1}(x-1)^n}{n}.$$

and

$$\ln x \approx (x-1) - \frac{(x-1)^2}{2} + \qquad + \frac{(-1)^{n-1}(x-1)^n}{n},$$

x near 1.