§ 6.3 Differential Equations

Suppose we have a car moving at a constant vel of 50 mph

Since for <u>constant velocity</u>

change in displacement = velocity × time,

we have

$$s = 50t.$$

Alternatively, since $v = \frac{ds}{dt}$, we could say

$$\frac{ds}{dt} = 50$$

Hence s is an antiderivative of 50. This means that

$$s = 50t + C$$

(where C is the value of s at $t=0$).

This is the basic idea behind <u>differential</u> <u>equations</u> - you're given an equation involving the derivative y' (and/or higher derivatives) of some function y and you want to solve for y.

Differential equations are very widespread in science especially physics.

Uniformly Accelerated Motion

This is the next simplest situation where an object travels with constant acceleration (but changing velocity).

A common example is an object falling under gravity. The acceleration due to gravity is

$$32 \, ft/s^2 \quad \text{or} \quad 9.81 \, m/s^2.$$

If we measure s, v upwards, then we have

$$a = \frac{dv}{dt} = -g$$

and we solve for v and maybe also s.

<u>Ex 1</u> A stone is dropped from a 100 ft high building. Find its position and vel. as fns of time. When does it hit the ground and how fast is it going at that time?

If we measure s upwards from the ground (and v upwards also), we have

$$\frac{dv}{dt} = -32 \text{ ft/s}^2$$

Taking antids., the general antid of -32 is $-32t + C$, so

$$v = -32t + C$$

At $t=0$, drop the stone which starts from rest. ie $v=0$ when $t=0$

So

$$0 = -32 \cdot 0 + C$$

$$\Rightarrow \quad C = 0$$

Thus

$$v = -32t.$$

In terms of displ., $v = \frac{ds}{dt}$ so

$$\frac{ds}{dt} = -32t \text{ ft/s}$$

Taking antids again, the general antid of $-32t$ is $-16t^2 + K$, K const so

$$s = -16t^2 + K$$

At $t=0$, the stone is 100ft off the ground, so $s = 100$

$$100 = -16 \cdot 0^2 + K$$

$\Rightarrow \quad k = 100.$

So

$$s = -16t^2 + 100 \text{ ft}$$

The stone hits the ground when $s = 0$, so ~~set~~

$$0 = -16t^2 + 100$$

$$16t^2 = 100$$

$$t^2 = \frac{100}{16}$$

$$t = \frac{10}{4} = 2.5 \text{ s.}$$

At this time the vel. is $-32 \times 2.5 = -80$ ft/sec.

($v < 0$ because the stone is moving downwards).

Ex 2. An egg is thrown vertically upward with a speed of 10 m/s from a height of 2m. Find the highest point it reaches and when it hits the ground.

Measure displ vertically upward from the ground.

$$\frac{dv}{dt} = -9.8$$

so taking antid's.

$$v = -9.8t + c$$

when $t = 0$, $v = 10$, so

$$10 = -9.8 \cdot 0 + c$$

$$\Rightarrow c = 10.$$

$$v = -9.8t + 10.$$

$v = \frac{ds}{dt}$ and so

$$\frac{ds}{dt} = -9.8t + 10$$

Take antids.

$$s = -4.9t^2 + 10t + K.$$

$s = 2$ when $t = 0$ so

$$2 = -4.9 \times 0^2 + 10 \times 0 + k$$

$$\rightarrow k = 2$$

and

$$s = -4.9t^2 + 10t + 2.$$

Egg reaches its highest pt when $v = 0$.

ie $$0 = -9.8t + 10$$

$$t = \frac{10}{9.8} \text{ s.} \sim 1.02 \text{ s.}$$

Here

$$s = -4.9(1.02)^2 + 10(1.02) + 2 \approx 7.10\,m.$$

Egg hits the ground when $s=0$.

ie

$$0 = -4.9t^2 + 10t + 2$$

Solving using quadratic formula gives

$$t \approx -.18,\ 2.22$$

Since we want $t>0$, the egg hits the ground at

$t \approx 2.22$ s.

More general differential equations.

To solve a differential equation of the form

$$\frac{dy}{dx} = f(x),$$

it suffices to find an antid. $F(x)$ of $f(x)$ and set

$$y = F(x) + C.$$

If in addition we want just one solution which has value y_0 at x_0, we need to pick the (unique) antid. which has value y_0 at x_0.

Problems of the form

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0$$

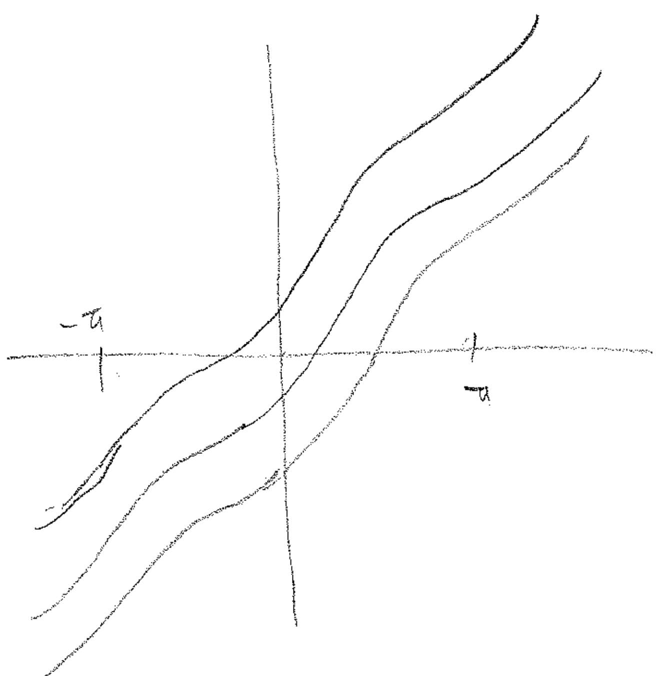
are called initial value problems.

Ex. a) Find the general sol^n of

$$\frac{dy}{dx} = \sin x + 2.$$

An antid. of $\sin x + 2$ is $-\cos x + 2x$,
so the general sol^n is

$$y = -\cos x + 2x + C.$$



Solve the IVP

$$\frac{dy}{dx} = \sin x + 2, \quad y(3) = 5$$

Want the antid.

$$y = -\cos x + 2x + c$$

to satisfy $y(3) = 5$, so

$$5 = -\cos 3 + 2 \times 3 + c$$

$$\Rightarrow c = 5 + \cos 3 - 6 \approx -1.99$$

Hence the (unique) sol^n is

$$y = -\cos x + 2x - 1.99$$