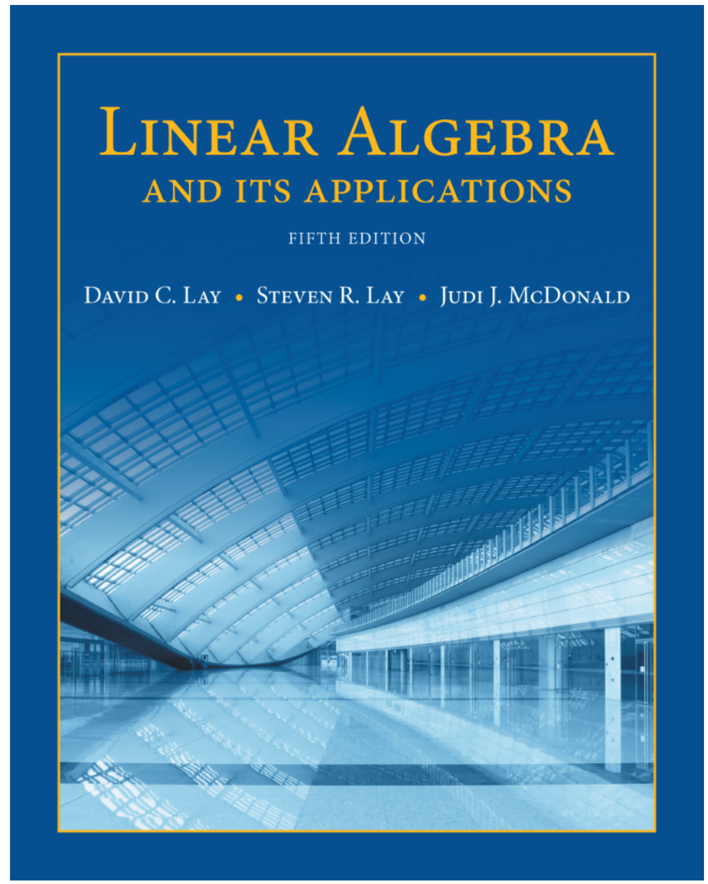
6 Orthogonality and Least Squares

6.1

INNER PRODUCT, LENGTH, DISTANCE, AND ANGLE BETWEEN VECTORS



INNER PRODUCT

- If **u** and **v** are vectors in $\mathbb{R}^n$, then we regard **u** and **v** as $n \times 1$ matrices.
- The transpose $\mathbf{u}^T$ is a $1 \times n$ matrix, and the matrix product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix, which we write as a single real number (a scalar) without brackets.
- The number $\mathbf{u}^T\mathbf{v}$ is called the **inner product** of **u** and **v**, and it is written as $u \cdot v$.
- This inner product is also referred to as a **dot product**.

INNER PRODUCT

- If $\mathrm{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathrm{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$,

  then the inner product of **u** and **v** is

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n .$$

THE LENGTH OF A VECTOR

- If **v** is in $\mathbb{R}^n$, with entries $v_1, \ldots, v_n$, then the square root of $v \cdot v$ is defined because $v \cdot v$ is nonnegative.

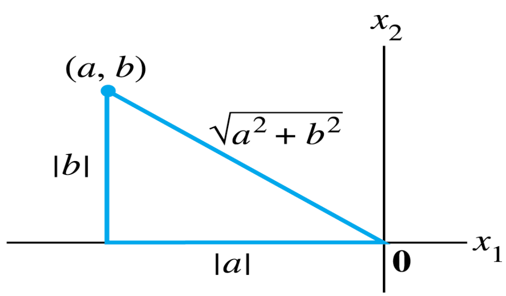
- **Definition:** The **length** (or **norm**) of **v** is the nonnegative scalar $\|\mathrm{v}\|$ defined by

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \text{ and } \|v\|^2 = \sqrt{v \cdot v}$$

- Suppose **v** is in $\mathbb{R}^2$, say, $\mathrm{v} = \begin{bmatrix} a \\ b \end{bmatrix}$

THE LENGTH OF A VECTOR

- If we identify **v** with a geometric point in the plane, as usual, then $\|\mathbf{v}\|$ coincides with the standard notion of the length of the line segment from the origin to **v**.
- This follows from the Pythagorean Theorem applied to a triangle such as the one shown in the following figure.



Interpretation of $\|\mathbf{v}\|$ as length.

- For any scalar c, the length $c\mathbf{v}$ is $|c|$ times the length of v. That is,

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$$

THE LENGTH OF A VECTOR

- A vector whose length is 1 is called a **unit vector**.
- If we *divide* a nonzero vector **v** by its length—that is, multiply by $1/\|\mathbf{v}\|$—we obtain a unit vector **u** because the length of **u** is $(1/\|\mathbf{v}\|)\|\mathbf{v}\|$.
- The process of creating **u** from **v** is sometimes called **normalizing v**, and we say that **u** is *in the same direction* as **v**.

THE LENGTH OF A VECTOR

- **Example 2:** Let $\mathrm{v} = (1, -2, 2, 0)$. Find a unit vector **u** in the same direction as **v**.
- **Solution:** First, compute the length of **v**:

$$\|\mathrm{v}\|^2 = v \cdot v = (1)^2 + (-2)^2 + (2)^2 + (0)^2 = 9$$

$$\|\mathrm{v}\| = \sqrt{9} = 3$$

- Then, multiply **v** by $1/\|\mathrm{v}\|$ to obtain

$$\mathrm{u} = \frac{1}{\|\mathrm{v}\|}\mathrm{v} = \frac{1}{3}\mathrm{v} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

DISTANCE IN $\mathbb{R}^n$

- To check that $\|\mathbf{u}\| = 1$, it suffices to show that $\|\mathbf{u}\|^2 = 1$.

$$\|\mathbf{u}\|^2 = u \cdot u = \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + (0)^2$$

$$= \frac{1}{9} + \frac{4}{9} + \frac{4}{9} + 0 = 1$$

- **Definition:** For **u** and **v** in $\mathbb{R}^n$, the **distance between u and v**, written as dist (**u**, **v**), is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$\text{dist}(\mathbf{u},\mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

DISTANCE IN $\mathbb{R}^n$

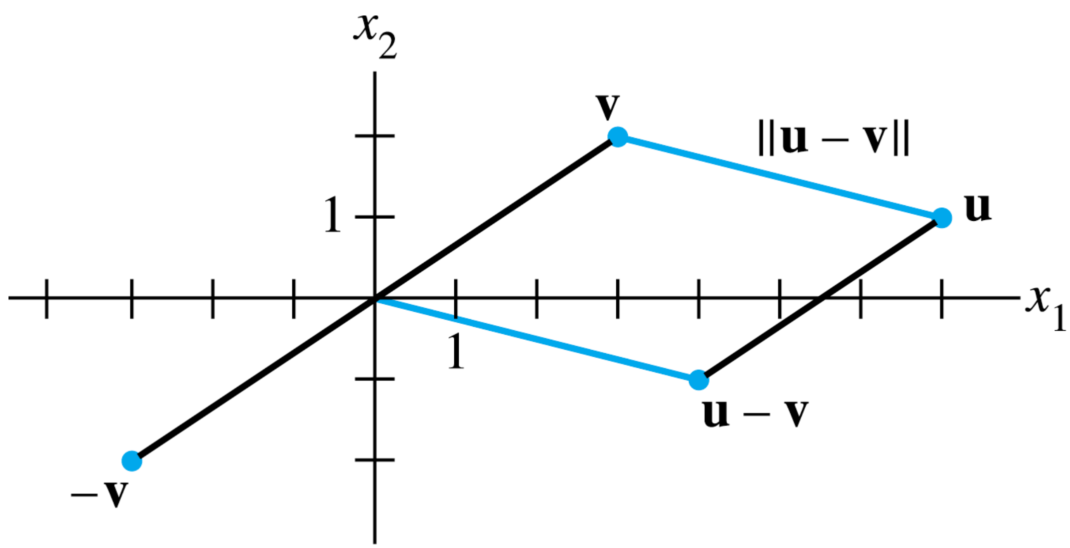
- **Example 4:** Compute the distance between the vectors $\mathbf{u} = (7,1)$ and $\mathbf{v} = (3,2)$.
- **Solution:** Calculate

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$$

- The vectors **u**, **v**, and $\mathbf{u} - \mathbf{v}$ are shown in the figure on the next slide.
- When the vector $\mathbf{u} - \mathbf{v}$ is added to **v**, the result is **u**.

DISTANCE IN $\mathbb{R}^n$



The distance between **u** and **v** is the length of **u** − **v**.

- Notice that the parallelogram in the above figure shows that the distance from **u** to **v** is the same as the distance from $\mathbf{u} - \mathbf{v}$ to **0**.

ANGLES IN $\mathbb{R}^2$ AND $\mathbb{R}^3$

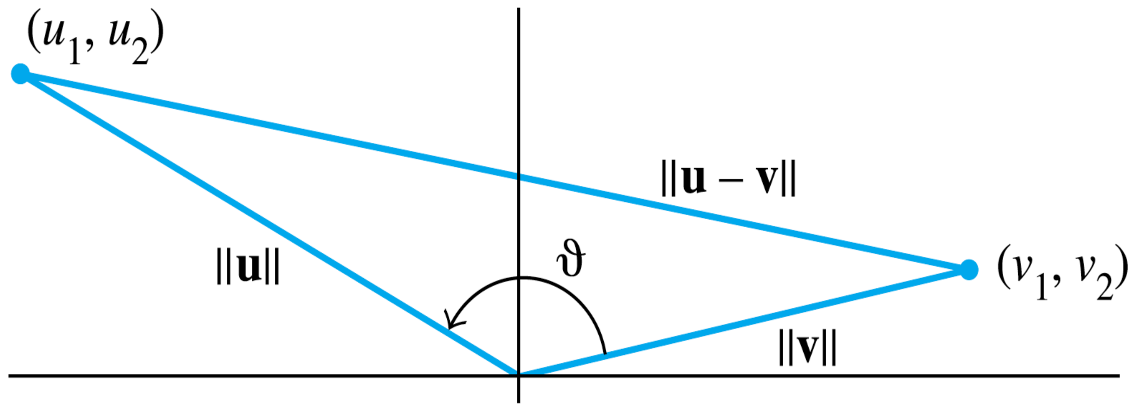
- If **u** and **v** are nonzero vectors in either $\mathbb{R}^2$ or $\mathbb{R}^3$, then there is a nice connection between their inner product and the angle ϑ between the two line segments from the origin to the points identified with **u** and **v**.

- The formula is

$$u \cdot v = \|\mathrm{u}\|\|\mathrm{v}\|\cos\vartheta \qquad (2)$$

- To verify this formula for vectors in $\mathbb{R}^2$, consider the triangle shown in the figure on the next slide with sides of lengths, $\|\mathrm{u}\|$, $\|\mathrm{v}\|$, and $\|\mathrm{u}-\mathrm{v}\|$.

ANGLES IN $\mathbb{R}^2$ AND $\mathbb{R}^3$



The angle between two vectors.

- By the law of cosines,

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\vartheta$$

which can be rearranged to produce the equations on the next slide.

ANGLES IN $\mathbb{R}^2$ AND $\mathbb{R}^3$

$$\|\mathrm{u}\|\|\mathrm{v}\|\cos\vartheta = \frac{1}{2}\left[\|\mathrm{u}\|^2 + \|\mathrm{v}\|^2 - \|\mathrm{u}-\mathrm{v}\|^2\right]$$

$$= \frac{1}{2}\left[u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1 - v_1)^2 - (u_2 - v_2)^2\right]$$

$$= u_1 v_1 + u_2 v_2$$

$$= u \cdot v$$

- The verification for $\mathbb{R}^3$ is similar.
- When $n > 3$, formula (2) may be used to *define* the angle between two vectors in $\mathbb{R}^n$.
- In statistics, the value of $\cos\vartheta$ defined by (2) for suitable vectors **u** and **v** is called a *correlation coefficient*.