Due 11:55pm Fri. Feb. 14 **NAME(S):** ____________________

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Math 215 Project 1 (25 pts) : Using Linear Algebra to solve GPS problem

1 Introduction

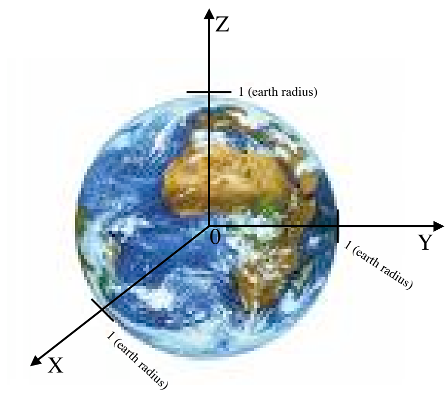
The age old question, *Where in the world am I?* can easily be solved nowadays by using the Global Positioning System (GPS). A GPS system consists of GPS satellites orbiting the earth and a GPS receiver on the earth's surface that receives signals from the satellites. A GPS receiver calculates its position by using the information sent by the GPS satellites. Each GPS satellite constantly transmits a signal at the speed of light containing the time the signal was sent and the location (i.e. orbit) of the GPS satellite sending the signal. In order for a GPS receiver to compute its position, it must *triangulate* its location from the satellites. To *triangulate*, a GPS receiver computes the distance from each satellite using the travel time of the radio signals sent. Since space has three dimensions this would seem to imply that only three satellites are required to compute the GPS receiver's location, but this is only true if the GPS receiver knows exactly when the signal is sent and received. This would require a super accurate (within nanoseconds) synchronized clock system between all of the GPS satellites and the GPS receiver. Not possible for any non-laboratory clock, making this an unusable system. Using four satellites instead of three, the GPS receiver can solve for time as well as position, eliminating the need for the GPS receiver to have a super accurate synchronized clock with the GPS satellites. However, this does not eliminate the need of a super accurate synchronized clock system between all of the GPS satellites. Each GPS satellite is equipped with an atomic clock , which is synchronized with all other GPS satellites and UTC (universal time coordinated) ignoring leap seconds. Atomic clocks are the most accurate timing instrument known, losing a maximum of only 1 second every 30,000 - 1,000,000 years and are accurate to within 10^{-9} seconds. Adjustments to the Atomic clocks on the GPS satellites are periodically update and synchronized from a control center on Earth. GPS time was set to zero on 0h 6-Jan-1980 and at this typing 9:26 a.m. September 8, 2015, the GPS time is week 1,861 and 221,248s. Since, four GPS satellites are required in order for a GPS receiver to accurately calculate its position, the Satellite Navigation system is set up in such way that from any point on Earth at least four satellites are always in line (i.e. visible) to a GPS receiver.

Since, the GPS receiver does not have a synchronized time clock with the satellites, the 4^{th} unknown variable in the system of equations is time, t. Ignoring atmospeheric conditions, the radio signals from the satellites travel to the receivers at the speed of light, c. Therefore, the distance to each satillites can be computed using the standard distance formula $d = c\cdot(t - t_{sent})$. Using the four satellites and the standard distance formula, we can set up a system of four equations with four unknowns. Solving this system for the unknowns yields the desired location of the GPS receiver.

1.1 GPS problem

Find the location of a GPS receiver we assume is on a boat at sea.

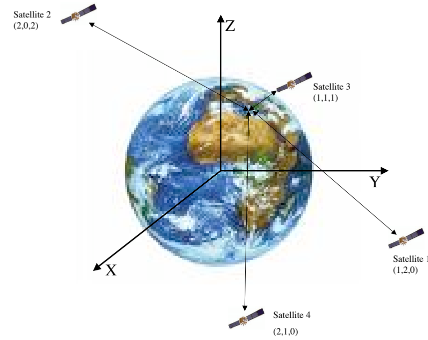
To simplify the GPS problem, we will use the Cartesian $xyz-$coordinate system with the earth centered at the origin, the positive z axis pointing through the north pole, and the unit of measure being the radius of the Earth. That is, each tick mark on an axis is equal to the radius of the earth.



We will also assume that any point at sea level satisfies $x^2+y^2+z^2=1$. Time will be measured in milliseconds, (one thousandth of a second, 10^{-3}) and the speed of light constant $c \approx 0.047$ earth radii per millisecond.

The GPS receiver on the boat receives from 4 satellites simultaneously, their position in relation to the xyz-coordinate system (units radius of earth) and the time the signal was sent in milliseconds. The data is given in the table below. The numbers are fabricated so that they are easy to manage, but they are not completely unrealistic.

Satellite	xyz-Position	Time signal was sent
1	(1,2,0)	19.9
2	(2,0,2)	2.4
3	(1,1,1)	32.6
4	(2,1,0)	19.9



Let (x, y, z) be the position of your GPS receiver on Earth and t be the time the signals arrive. For Satellite 1, we have

$$d = 0.047 \cdot (t - 19.9)$$

and using the Euclidean distance formula for points in space we have

$$d = \sqrt{(x-1)^2 + (y-2)^2 + (z-0)^2}.$$

Now combine the two equations together to get

$$\begin{aligned} \sqrt{(x-1)^2 + (y-2)^2 + (z-0)^2} &= 0.047 \cdot (t - 19.9) \\ (x-1)^2 + (y-2)^2 + (z-0)^2 &= 0.047^2 \cdot (t - 19.9)^2 \leftarrow \text{obtained by squaring both sides} \end{aligned}$$

Expand all squares and rearrange so that only linear variables are on the left side.

$$\text{(E1)} \quad 2x + 4y + 0z - 2(0.047)^2(19.9)t = 1^2 + 2^2 - 0.047^2(19.9)^2 + x^2 + y^2 + z^2 - 0.047^2t^2$$

EXERCISE 1 (5pts):
Derive 3 more equations (E2), (E3), (E4) similar to (E1) for the other 3 satellites. Write all equations in the same format, i.e. only linear variables on the left side of the equal sign as in the above equation (E1).

EXERCISE 2 (5pts):
You should notice that all four equations have the same quadratic term on the right side, $x^2 + y^2 + z^2 - 0.047^2t^2$. Use equation (E1) to eliminate the quadratic terms, hence creating a linear system of 3 equations (L1), (L2), and (L3) and 4 unknowns x, y, z, and t. This can be done by subtracting (E1) from each equation, i.e. (L1) = (E2) - (E1), (L2) = (E3) - (E1), and (L3) = (E4) - (E1).

EXERCISE 3 (5pts):
Write out your linear equations (L1), (L2), and (L3) in augmented matrix form and solve for x, y, z, t using Octave, Matlab, or Mathematica. Note that there are 3 equations and 4 unknowns. Let t be the free variable and write the variables x, y, z in terms of t.

EXERCISE 4 (5pts):
Plug your variables x, y, z from Exercise 3 into (E1) and solve for t. Notice you should end up with a quadratic in terms of t and you should have two time solutions. Determine which time solution is correct by using the formula $x^2 + y^2 + z^2 = 1$ and report your final answer, the location of the GPS receiver in terms of the Cartesian (x, y, z).

Your answer in exercise 4 for the location of the GPS receiver is given with respect to the Cartesian xyz-coordinate system and would need to be converted into latitude and longitude in order to be useful. The xyz-coordinate system can be converted into ellipsoidal coordinates (ϕ, λ, h) where ϕ corresponds to latitude and λ to longitude and h to the ellipsoidal height. We will ignore h since we already know that are GPS receiver is at sea level. An ellipsoidal coordinates system is used since the Earth is not a perfect sphere. The size and shape of an ellipsoidal can be defined by its semi-major axis a and its semi-minor axis b. The semi-major axis a is the distance from the center of the Earth to the equator and the semi-minor axis b is the distance from the center of the Earth to the North pole. For our calculations we will use the Airy 1830 ellipsoidal to represent the Earth, this gives $a = 6,377,563.396$m and $b = 6,356,256.910$m.

The longitude λ conversion is very easy (remember the location of the quadrant of the angle),

$$\lambda = \arctan(\frac{y}{x})$$

However, the latitude ϕ conversion requires an iterative procedure,

$$\begin{aligned}
\phi_0 &= \arctan(\tfrac{z}{p(1-e)}) \\
v_0 &= \tfrac{a}{\sqrt{1-e\sin^2(\phi_0)}} \\
\phi_1 &= \arctan(\tfrac{z+ev_0\sin(\phi_0)}{p}) \\
v_1 &= \tfrac{a}{\sqrt{1-e\sin^2(\phi_1)}} \\
\phi_2 &= \arctan(\tfrac{z+ev_1\sin(\phi_1)}{p}) \\
v_2 &= \tfrac{a}{\sqrt{1-e\sin^2(\phi_2)}} \\
\phi_3 &= \arctan(\tfrac{z+ev_2\sin(\phi_2)}{p})
\end{aligned}$$

where $e = \frac{a^2-b^2}{a^2} = 6.6705397616 \cdot 10^{-3}$, and $p = \sqrt{x^2+y^2}$.
Your final answer for latitude is ϕ_3.

EXERCISE 5 (5pts):
Convert your xyz-coordinate answer first into meters (using the fact that the radius of earth is 6,377,563.396 meters) then into longitude λ and latitude ϕ_3. Use Octave, Matlab, or Mathematica to run the iterations. Do NOT do these calculations by hand or with a calculator!

SUBMISSION OF PROJECT
Do NOT email the project. Submit the project using Sakai. In the course shell, within Sakai, there is a link on the left side table called Assignment Tool. Click on Assignment tool and upload your project. If you are doing this project as a group make sure all names appear your submission. ALL members of the group are required to upload the project in Assignment Tool.

These calculations give you an insight on the foundation of a GPS system. However, this is not exactly how a GPS system works. There are many missing pieces. For instance, the speed of the radio signal is not constant due to atmospheric conditions, the satellite positions are only known to an accuracy of 1 to 3 meters, and the times the signals are sent have errors. Note that if an atomic clock has an error of only 10 nanoseconds (10^{-8} seconds) this will create a positioning error of about 3 meters. Current GPS receivers resolves these errors through a more advance mathematical system that produces a usable positioning unit. See Gilbert Strang and Kai Borre's book called *Linear Algebra , Geodesy, and GPS* and *Essentials of Satellite Navigation* for more details.