MATH 215
Practice -Exam 2

1. Find the inverse of the given matrices. Show **ALL** row operations that you used.

a) $A = \begin{bmatrix} 4 & -3 \\ 8 & -1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 4 & -3 & 1 & 0 \\ 8 & -1 & 0 & 1 \end{array}\right]$$

$-2R_1 + R_2 \to R_2$

$$\left[\begin{array}{cc|cc} 4 & -3 & 1 & 0 \\ 0 & 5 & -2 & 1 \end{array}\right]$$

$\frac{1}{5}R_2 \to R_2$

$$\left[\begin{array}{cc|cc} 4 & -3 & 1 & 0 \\ 0 & 1 & -2/5 & 1/5 \end{array}\right]$$

$3R_2 + R_1 \to R_1$

$$\left[\begin{array}{cc|cc} 4 & 0 & -1/5 & 3/5 \\ 0 & 1 & -2/5 & 1/5 \end{array}\right]$$

$\frac{1}{4}R_1 \to R_1$

$$\left[\begin{array}{cc|cc} 1 & 0 & -1/20 & 3/20 \\ 0 & 1 & -2/5 & 1/5 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1/20 & 3/20 \\ -2/5 & 1/5 \end{bmatrix}$$

b) $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 2 & 2 & 4 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 4 & 0 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$-3R_1 + R_2 \to R_2$
$-2R_1 + R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -5 & 1 & -3 & 1 & 0 \\ 0 & -2 & 2 & -2 & 0 & 1 \end{array}\right]$$

$\frac{1}{2}R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -5 & 1 & -3 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1/2 \end{array}\right]$$

$-R_3 + R_2 \to R_2$
$-R_3 + R_1 \to R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 2 & 0 & -1/2 \\ 0 & -4 & 0 & -2 & 1 & -1/2 \\ 0 & -1 & 1 & -1 & 0 & 1/2 \end{array}\right]$$

$-\frac{1}{4}R_2 \to R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 2 & 0 & -1/2 \\ 0 & 1 & 0 & 1/2 & -1/4 & 1/8 \\ 0 & -1 & 1 & -1 & 0 & 1/2 \end{array}\right]$$

$-3R_2 + R_1 \to R_1$
$R_2 + R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 3/4 & -7/8 \\ 0 & 1 & 0 & 1/2 & -1/4 & 1/8 \\ 0 & 0 & 1 & -1/2 & -1/4 & 5/8 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1/2 & 3/4 & -7/8 \\ 1/2 & -1/4 & 1/8 \\ -1/2 & -1/4 & 5/8 \end{bmatrix}$$

c) Using the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 2 & 2 & 4 \end{bmatrix}$ from part b) above solve $Ax = b$, where $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$$Ax = b$$

$$X = A^{-1}b$$

$$X = \begin{bmatrix} 1/2 & 3/4 & -7/8 \\ 1/2 & -1/4 & 1/8 \\ -1/2 & -1/4 & 5/8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 - 14/8 \\ 1/2 + 2/8 \\ -1/2 + 10/8 \end{bmatrix} = \begin{bmatrix} -5/4 \\ 3/4 \\ 3/4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -5 \\ 2 & 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 7 & -1 \\ 2 & 9 & 7 \end{bmatrix}$$

2. Using **cofactor expansion across the first row** to compute the determinant of A.

$$\det(A) = (-1)^{1+1} \cdot 1 \cdot \det\begin{bmatrix} -3 & -5 \\ 1 & -1 \end{bmatrix} + (-1)^{1+2} \cdot 2 \cdot \det\begin{pmatrix} -2 & -5 \\ 2 & -1 \end{pmatrix} + (-1)^{1+3} \cdot 4 \cdot \det\begin{pmatrix} -2 & -3 \\ 2 & 1 \end{pmatrix}$$
$$= (3+5) - 2 \cdot (2+10) + 4 \cdot (-2+6)$$
$$= 0$$

3. Using **cofactor expansion down last column** to compute the determinant of B.

$$\det(B) = (-1)^{1+3} \cdot (-2) \cdot \det\begin{bmatrix} 2 & 7 \\ 2 & 9 \end{bmatrix} + (-1)^{2+3} \cdot (-1) \cdot \det\begin{pmatrix} 1 & 4 \\ 2 & 9 \end{pmatrix} + (-1)^{3+3} \cdot 7 \cdot \det\begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}$$
$$= -2 \cdot (18-14) + (9-8) + 7 \cdot (7-8)$$
$$= -14$$

4. Using row operations combined with cofactor expansion, compute the determinant of A

$$\det(A) = \det\begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -5 \\ 2 & 1 & -1 \end{bmatrix} \underset{\substack{2R_1+R_2 \to R_2 \\ -2R_1+R_3 \to R_3}}{=} \det\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & -3 & -9 \end{bmatrix} \underset{3R_2+R_3 \to R_3}{=} \det\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = 1 \cdot 1 \cdot 0 = 0$$

5. Using row operations combined with cofactor expansion, compute the determinant of B

$$\det(B) = \det\begin{bmatrix} 1 & 4 & -2 \\ 2 & 7 & -1 \\ 2 & 9 & 7 \end{bmatrix} \underset{\substack{-2R_1+R_2 \to R_2 \\ -2R_1+R_3 \to R_3}}{=} \det\begin{bmatrix} 1 & 4 & -2 \\ 0 & -1 & 3 \\ 0 & 1 & 11 \end{bmatrix} = (-1)^{1+1} \cdot (1) \cdot \det\begin{bmatrix} -1 & 3 \\ 1 & 11 \end{bmatrix}$$
$$= (-1) \cdot 11 - 3 \cdot 1 = -14$$

6. What is the determinant of AB? What is the determinant of A^T?

$$\det(AB) = \det(A) \cdot \det(B) = 0 \cdot (-14) = 0$$
$$\det(A) = \det(A^T) = 0$$

7. Matrix A invertible? Matrix B invertible? Do the columns of A span $\mathbf{R}^3$?
Are the columns of B linearly independent?

$\det(A) = 0 \Rightarrow$ A is not invertible (a.k.a singular)

$\det(B) = -14 \Rightarrow$ A is invertible (a.k.a. non-singular)

$\det(A) = 0 \Rightarrow$ columns of A do not span $\mathbb{R}^3$.

$\det(B) = -14 \Rightarrow$ columns of B are linearly independent

Reference Section 2.3 Invertible Matrix theorem.

(Same matrices A and B from page 2)

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -5 \\ 2 & 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 7 & -1 \\ 2 & 9 & 7 \end{bmatrix}$$

8. Compute the product AB.

$$A \cdot B = \begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -5 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & -2 \\ 2 & 7 & -1 \\ 2 & 9 & 7 \end{bmatrix} = \begin{bmatrix} 13 & 54 & 24 \\ -18 & -74 & -28 \\ 2 & 6 & -12 \end{bmatrix}$$

Should be able to do this calculation without a calculator.

9. Compute A^T and B^T and the product $(AB)^T$.

$$A^T = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & 1 \\ 4 & -5 & -1 \end{bmatrix} \quad B^T = \begin{bmatrix} 1 & 2 & 2 \\ 4 & 7 & 9 \\ -2 & -1 & 7 \end{bmatrix} \quad (AB)^T = B^T \cdot A^T = \begin{bmatrix} 13 & -18 & 2 \\ 54 & -74 & 6 \\ 24 & -28 & -12 \end{bmatrix}$$

10. (Section 6.1) Let $u = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$.

Compute $u^T u$, uu^T, $u^T v$, $v^T u$, vu^T, uv^T, $\|u\|$, $\|v\|$, and the angle between u and v.

$$u^T \cdot u = (-3\ 0\ 1)\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = 9 + 0 + 1 = 10$$

$$u \cdot u^T = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \cdot (-3\ 0\ 1) = \begin{bmatrix} 9 & 0 & -3 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$u^T \cdot v = (-3\ 0\ 1) \cdot \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = -6 + 0 + 4 = -2$$

$$v^T \cdot u = (2\ 1\ 4) \cdot \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = -6 + 0 + 4 = -2$$

$$v \cdot u^T = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \cdot (-3\ 0\ 1) = \begin{bmatrix} -6 & 0 & 2 \\ -3 & 0 & 1 \\ -12 & 0 & 4 \end{bmatrix}$$

$$u \cdot v^T = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \cdot (2\ 1\ 4) = \begin{bmatrix} -6 & -3 & -12 \\ 0 & 0 & 0 \\ 2 & 1 & 4 \end{bmatrix}$$

$$\|u\| = \sqrt{u^T \cdot u} = \sqrt{10}$$

$$\|v\| = \sqrt{v^T \cdot v} = \sqrt{(2\ 1\ 4)\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}} = \sqrt{4 + 1 + 16} = \sqrt{21}$$

$$\cos\theta = \frac{u^T \cdot v}{\|u\| \cdot \|v\|} = \frac{-2}{\sqrt{10} \cdot \sqrt{21}}$$

$$\cos\theta = \frac{-2}{\sqrt{210}}$$

$$\theta = \arccos\left(\frac{-2}{\sqrt{210}}\right)$$

$= 1.71$ rads or 98°