**Example 1:** Given the following linear system. Use row operations to find the solution set.

$$3x_2 - 6x_3 + 6x_4 + 4x_5 = -5$$
$$3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9$$
$$6x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15$$

**Step 1:** Represent the system of equations as an augmented matrix.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 6 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

**Step 2:** Reduced the system into an equivalent "much" easier to solve system. Start by getting the augmented matrix into row echelon form, i.e. sort of "upper triangular".

Row swap to get a non-zero value in the (1,1)-entry of your augmented matrix

1.) $R_1 <-> R_2$

$$\begin{bmatrix} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 6 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Zero out the (3,1)-entry of your augmented matrix.

2.) $-2R_1 + R_3 -> R_3$

$$\begin{bmatrix} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 5 & -4 & 1 & -10 & -3 \end{bmatrix}$$

Using the (2,2)-entry of your augmented matrix zero out the (3,2)-entry. Remember the goal is to get the matrix into row echelon form.

3.) $-\frac{5}{3}R_2 + R_3 -> R_3$

$$\begin{bmatrix} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 0 & 6 & -9 & \frac{-50}{3} & \frac{16}{3} \end{bmatrix}$$

This system is in row echelon form. Now transform the augmented matrix representation back into a linear system of equations.

$$3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9$$
$$3x_2 - 6x_3 + 6x_4 + 4x_5 = -5$$
$$6x_3 - 9x_4 - \frac{50}{3}x_5 = 15$$

At this point, you should notice that the linear system does not have a unique solution. You could find a solution by picking values for $x_4$ and $x_5$, then solve for $x_3$ in equation 3. And proceed to get values for $x_1$ and $x_2$ by using equations 1 and 2. However, in order to represent ALL infinite solutions, we still need to do some work, i.e. get the augmented matrix into reduced row echelon form, then transform the augmented matrix into a linear system, and finally write out the solution set. Continuing

$$\begin{bmatrix} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 0 & 6 & -9 & \frac{-50}{3} & \frac{16}{3} \end{bmatrix}$$

Lets proceed by making the leading non-zero coefficient in each row a 1.

4.) $\frac{1}{3}R_1 -> R_1$

5.) $\frac{1}{3}R_2 -> R_2$

6.) $\frac{1}{6}R_3 -> R_3$

$$\begin{bmatrix} 1 & \frac{-7}{3} & \frac{8}{3} & \frac{-5}{3} & \frac{8}{3} & 3 \\ 0 & 1 & -2 & 2 & \frac{4}{3} & \frac{-5}{3} \\ 0 & 0 & 1 & \frac{-3}{2} & \frac{-25}{9} & \frac{8}{9} \end{bmatrix}$$

Zero out the (1,2)-entry using the (2,2) entry.

7.) $\frac{7}{3}R_2 + R_1 -> R_1$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & \frac{52}{9} & \frac{-8}{9} \\ 0 & 1 & -2 & 2 & \frac{4}{3} & \frac{-5}{3} \\ 0 & 0 & 1 & \frac{-3}{2} & \frac{-25}{9} & \frac{8}{9} \end{bmatrix}$$

Zero out the (1,3)-entry and the (2,3) entry using the (3,3)-entry.

8.) $2R_3 + R_1 \to R_1$

9.) $2R_3 + R_2 \to R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{2}{9} & \frac{8}{9} \\ 0 & 1 & 0 & -1 & \frac{-38}{9} & \frac{1}{9} \\ 0 & 0 & 1 & \frac{-3}{2} & \frac{-25}{9} & \frac{8}{9} \end{bmatrix}$$

The augmented matrix is in reduced row echelon form. Now represent augmented matrix as a linear system.

$$\begin{aligned} x_1 \quad\quad + \frac{2}{9}x_5 &= \frac{8}{9} \\ x_2 \quad - x_4 - \frac{38}{9}x_5 &= \frac{1}{9} \\ x_3 - \frac{3}{2}x_4 - \frac{25}{9}x_5 &= \frac{8}{9} \end{aligned}$$

**Step 3:** The variables associated with the leading 1's in the augmented matrix ($x_1, x_2, x_3$) are called the <u>leading variables</u>. All other variables ($x_4, x_5$) are called <u>free variables</u>. NOTE: that each leading variable only occurs in one equation. The reason is that the reduced row echelon form causes this to happen.

Solve for the leading variables and set the free variables equal to themselves.

$$\begin{aligned} x_1 &= \frac{8}{9} - \frac{2}{9}x_5 \\ x_2 &= \frac{1}{9} + x_4 + \frac{38}{9}x_5 \\ x_3 &= \frac{8}{9} + \frac{3}{2}x_4 + \frac{25}{9}x_5 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned}$$

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<u>Example 2:</u> Consider the following linear system. Use row operations to find the solution set.

$$\begin{aligned} 12x_1 - 7x_2 + 11x_3 - 9x_4 + 5x_5 &= 7 \\ -9x_1 + 4x_2 - 8x_3 + 7x_4 - 3x_5 &= -5 \\ -6x_1 + 11x_2 - 7x_3 + 3x_4 - 9x_5 &= 8 \end{aligned}$$

**Step 1:** Represent the system of equations as an augmented matrix.

$$AUG := \begin{bmatrix} 12 & -7 & 11 & -9 & 5 & 7 \\ -9 & 4 & -8 & 7 & -3 & -5 \\ -6 & 11 & -7 & 3 & -9 & 8 \end{bmatrix}$$

**Step 2:** Reduced the augmented matrix into reduced row echelon form using row operations. The following is the reduced row echelon form of the matrix.

$$\begin{bmatrix} 1 & 0 & \frac{4}{5} & \frac{-13}{15} & 0 & \frac{9}{10} \\ 0 & 1 & \frac{-1}{5} & \frac{-1}{5} & 0 & \frac{-41}{10} \\ 0 & 0 & 0 & 0 & 1 & \frac{-13}{2} \end{bmatrix}$$

Now represent augmented matrix as a linear system.

$$\begin{aligned} x_1 \quad + \frac{4}{5}x_3 - \frac{13}{15}x_4 \quad &= \frac{9}{10} \\ x_2 - \frac{1}{5}x_3 - \frac{1}{5}x_4 \quad &= \frac{-41}{10} \\ x_5 &= \frac{-13}{2} \end{aligned}$$

**Step 3:** The variables associated with the leading 1's in the augmented matrix ($x_1, x_2, x_5$) are called the <u>leading variables</u>. All other variables ($x_3, x_4$) are called <u>free variables</u>. Solve for the leading variables and set the free variables equal to themselves.

$$\begin{aligned} x_1 &= \frac{9}{10} - \frac{4}{5}x_3 + \frac{13}{15}x_4 \\ x_2 &= -\frac{41}{10} + \frac{1}{5}x_3 + \frac{1}{5}x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \\ x_5 &= -\frac{13}{2} \end{aligned}$$