SELF-MAPS OF $\mathbb{P}^2$ WITH INVARIANT ELLIPTIC CURVES

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Abstract. We discuss the geometric and dynamical properties of the holomorphic self-maps of $\mathbb{P}^2$ that leave invariant an elliptic plane curve.

1. Introduction.

Given an elliptic plane curve $Q$, we consider the problem of constructing holomorphic self-maps $f$ of $\mathbb{P}^2$ that leave $Q$ invariant. In section 2, we state the criterion for a self-map of $Q$ to extend to $\mathbb{P}^2$. We look in section 3 at the singular points of $Q$. In contrast with the smooth case, most singular elliptic curves do not admit non-trivial self-maps. The obstructions given by the singular points of $Q$ are discussed in section 3. We define two invariants, in terms of Weierstrass’ $\sigma$ and $\zeta$ functions, and state an invariance criterion for the elliptic plane curves with ordinary singularities.

In section 4, we prove that do not exist self-maps of $\mathbb{P}^2$, for which $Q$ is critical and invariant.

We prove in section 5 that the backward orbit of any point of $Q$ is dense in the Julia set of $f$.

In section 6 we discuss the case when $Q$ is a smooth cubic. The classic tangent process on $Q$ provides examples of self-maps that leave $Q$ invariant. If we require $f$ to leave invariant a line of lines, $Q$ must be isomorphic to the Fermat cubic. We also discuss in this section the case when $f$ has minimal degree 2.

When an elliptic plane curve has enough symmetries, the invariants associated to its singular points can be calculated easily. The simplest case is the dual of a smooth cubic (section 7). In section 8, we consider special families of elliptic quartics with two singular points. Computer-generated pictures illustrate tangent processes on such curves.

2. Preliminaries.

Let $\mathbb{C}_d$ denote the space of homogeneous polynomials of degree $d$ in three variables. A rational self-map $f$ of $\mathbb{P}^2$, of algebraic degree $d(f) = d$ is given by three polynomials $p_0, p_1, p_2$ in $\mathbb{C}_d$ with no common divisors, according to the formula $f[x_0, x_1, x_2] = [p_0(x_0, x_1, x_2), p_1(x_0, x_1, x_2), p_2(x_0, x_1, x_2)].$ Let $I(f)$ denote the set of indeterminacy of $f$, formed by the common zeroes in $\mathbb{P}^2$ of the polynomials $p_j$.

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When $I(f) = \emptyset$, $f$ is regular (holomorphic). We denote by $\deg(f)$ the topological degree of a map $f$. When $\mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2$ is regular, $\deg(f) = d(f)^2$.

Every effective divisor $D$ on $\mathbb{P}^2$ of degree $\deg(D) = e$ is given by an equation $(p = 0)$, with $0 \neq p \in \mathbb{C}$ determined by $D$ up to a multiplicative constant. We denote by $\simeq$ the linear equivalence of divisors, and also the isomorphism of line bundles, two divisors on $\mathbb{P}^2$ are linearly equivalent if they have the same degree. The pull-back of a divisor $D$ through a map $f$ is denoted by $f^* D$. Given a rational self-map $f$ of $\mathbb{P}^2$ and a divisor $D = (p = 0)$ on $\mathbb{P}^2$ whose support does not contain the image of $f$, $f^* D = (p(f) = 0)$, with $\deg(f^* D) = d(f) \deg(D)$. Let $\mathcal{O}_{\mathbb{P}^2}(d)$ denote the line bundle on $\mathbb{P}^2$ whose global holomorphic sections vanish on effective divisors of degree $d$. Its restriction to a plane curve $Q$ is denoted $\mathcal{O}_Q(d)$. (By “curve”, we mean a one-dimensional irreducible variety.)

**Remark 2.1.** Let $f$ be a non-constant rational self-map of $\mathbb{P}^2$. If $Q$ is plane curve with $I(f) \cap Q = \emptyset$, then $C := f(Q)$ is a curve. If $\overline{f} : C \to \mathbb{P}^2$ denotes the restriction of $f$ to $Q$, then $g^* \mathcal{O}_C(1) \simeq \mathcal{O}_{\mathbb{P}^2}(d(f))$, hence $\frac{\deg(g)}{\deg(f)} = \frac{\deg(f)}{\deg(C)}$.

**Proof.** Since $I(f) \cap Q = \emptyset$, $Q$ is not contracted by $f$. Let $\overline{f} : C \to \mathbb{P}^2$ be the embedding map. By Bertini, the pull-back $f^* I$ of the generic line $I$ in $\mathbb{P}^2$ meets $Q$ transversely, and its trace on $Q$ equals $(i g)^* I$. We get $g^* \mathcal{O}_C(1) \simeq (i g)^* \mathcal{O}_{\mathbb{P}^2}(1) \simeq \mathcal{O}_{\mathbb{P}^2}(d(f))$.

**Definition 2.2.** A plane curve $Q$ is invariant for a rational self-map $f$ of $\mathbb{P}^2$ iff $I(f) \cap Q = \emptyset$ and $f(Q) = Q$. Given a regular self-map $g$ of $Q$, a regular (resp. rational) extension of $g$ to $\mathbb{P}^2$ is a regular (resp. rational) self-map $f$ of $\mathbb{P}^2$ that leaves $Q$ invariant and satisfies $f|_Q = g$.

Given a plane curve $Q$, the degree $d$ divisors on $\mathbb{P}^2$ cut out on $Q$ a complete linear system. This yields a criterion for a self-map of $Q$ to extend to $\mathbb{P}^2$.

**Proposition 2.3.** Given a plane curve $Q$ and an integer $d \geq \deg(Q)$, a self-map $Q \xrightarrow{f} \mathbb{P}^2$ has regular extensions $\mathbb{P}^2 \xrightarrow{\overline{f}} \mathbb{P}^2$ with $d(f) = d$ iff $g^* \mathcal{O}_Q(1) \simeq \mathcal{O}_{\mathbb{P}^2}(d)$.

**Proof.** Write $Q = (q = 0)$, with $q \in \mathbb{C}$, $e = \deg(Q)$. Let $i = [s_0, s_1, s_2]$ denote the embedding of $Q$ in $\mathbb{P}^2$, where $s_j \in \Gamma(Q, \mathcal{O}_Q(1))$ are global sections in $\mathcal{O}_Q(1)$. Then $i g = [g^* s_0, g^* s_1, g^* s_2]$, $g^* s_j \in \Gamma(Q, g^* \mathcal{O}_Q(1))$. The map $\mathcal{C}_d \xrightarrow{\overline{f}} \Gamma(Q, g^* \mathcal{O}_Q(d))$, $\beta(p) = p(s_0, s_1, s_2)$ is an epimorphism. For all $c \in Q$, $\beta(p)(c) = 0$ if $p(c) = 0$. Consequently, $p \in \ker(\beta)$ iff $q$ divides $p$.

If $g^* \mathcal{O}_Q(1) \simeq \mathcal{O}_Q(d)$, there exist $p_j \in C_d$ with $g^* s_j = \beta(p_j)$ for all $c \in Q$, $p_j(c) = 0$ if $s_j(g(c)) = 0$. Since $s_0, s_1, s_2$ have no common zeros, $p_0, p_1, p_2$ have no common zeros on $Q$. Since $p_0, p_1, p_2$ and $q$ have no common zeros, for generic $r_j \in \mathbb{C} - \{c\}$, $f := [p_0 + q r_0, p_1 + q r_1, p_2 + q r_2]$ is a regular self-map of $\mathbb{P}^2$. We have $i g = [g^* s_0, g^* s_1, g^* s_2] = [\beta(p_0), \beta(p_1), \beta(p_2)] = [\beta(p_0 + q r_0), \beta(p_1 + q r_1), \beta(p_2 + q r_2)] = [p_0 + q r_0, p_1 + q r_1, p_2 + q r_2] \circ [s_0, s_1, s_2] = f_i$.

If $\mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2$ satisfies $f|_Q = g$, then, by Remark 2.1, $g^* \mathcal{O}_Q(1) \simeq \mathcal{O}_Q(d(f))$.

**Remark 2.4.** Given any integer $d > 0$, $Q \xrightarrow{g} \mathbb{P}^2$ has rational extensions $f$ with $d(f) = d$ iff $g^* \mathcal{O}_Q(1) \simeq \mathcal{O}_Q(d)$. Assuming this, let $e = \deg(Q)$. When $d < e$, $g$ has a unique rational extension $f$ with $d(f) = d$. When $d \geq e$, the regular extensions of $g$ with $d(f) = d$ form a Zariski open subset of $\mathbb{C}^N$, with $N = \binom{d - e + 2}{2}$. 

Proof. The first statement follows from the proof of Proposition 2.3. When \( d < e \), the evaluation map \( \beta \) is an isomorphism. When \( d \geq e \), \( \dim(\ker(\beta)) = N \). Now, let \( f = [p_0, p_1, p_2] \) and \( \hat{f} = [\hat{p}_0, \hat{p}_1, \hat{p}_2] \) be rational extensions of \( g \) with \( d(f) = d = d(\hat{f}) \). The map \( Q \to \mathbb{P}_1, k = \frac{\partial}{\partial f} \), is independent of \( j \), hence it does not have zeros or poles, i.e. it is constant. We may assume \( k = 1 \), and then \( \hat{p}_j - p_j \in \ker(\beta) \). \( \square \)

Remark 2.5. Given a plane curve \( Q \), an integer \( d > \deg(Q) \), a map \( \mathbb{P}_2 \to Q \) with \( g^* \mathcal{O}_Q(1) \cong d \mathcal{O}_Q(1) \), and a point \( a \in \mathbb{P}_2 \setminus Q \), there exist regular extensions of \( g \) to \( \mathbb{P}_2 \) for which \( a \) is an attracting fixed point.

Proof. Write \( Q = (q = 0) \), with \( q \in \mathbb{C} \). We may assume that \( a = [1, 0, 0] \) and \( q(1, 0, 0) = 1 \). Fix a rational extension of \( g \), \( \hat{f} = [p_0, p_1, p_2] \), with \( d(\hat{f}) = d \). Take \( m \in \mathbb{C} \) with \( m(1, 0, 0) = 0 \) so that \( p_1 \) and \( p_2 \) have no common factors, where \( p_j = \hat{p}_j - p_j(1, 0, 0)x_0^{-d} - \frac{m^{d-q}}{q} \). Let \( B \subset \mathbb{P}_1 \) denote the finite set of common zeros of \( p_1 \) and \( p_2 \). Note that \( a \in B \). Take \( l \in \mathbb{C} \) so that \( l(1, 0, 0) = 1 \) and \( l(b) \neq 0 \) for all \( b \in B \). For \( k \in \mathbb{C} \), put \( \hat{p}_0 = \hat{p}_0 - kl^{d-\varepsilon q} \), and define \( f_k = [p_0, p_1, p_2] \). If \( b \in I(f_k) \), then \( b \in B \), \( q(b) \neq 0 \) and \( k = \hat{p}_0(b)/(l^{d-\varepsilon}(b)q(b)) \). Therefore, for all but finitely many values of \( k \), \( f_k \) is regular. Clearly, \( f_k(a) = a \). For large \( k \), \( a \) is an attracting fixed point of \( f_k \). \( \square \)

A curve \( Q \) with self-maps of degree greater than 1 must be rational or elliptic. Given a rational plane curve \( Q \), it is easy to find regular self-maps \( f \) of \( \mathbb{P}_2 \) for which \( Q \) is invariant and critical. In this paper we discuss the case when \( Q \) is elliptic.

Fix a normalization map \( C \to Q \), and a group structure \( C \to (C, +, 0) \). Given \( Q \to Q \), let \( C \to C \) be its lifting through \( \nu \), \( \nu h = g \). There exist \( m, n \in \mathbb{C} \) such that \( h[t] = [mt - n] \) for all \( t \in \mathbb{C} \). The multiplier \( m(g) \) \( \nu \) does not depend on \( \nu \) or \( [\cdot] \), and \( \deg(g) = |m|^2 \). Let \( \mathbb{Z}(C) \) denote the ring formed by the multipliers of the self-maps of \( C \), and \( \mathbb{U}(C) \) the group of units in \( \mathbb{Z}(C) \). Given a rational self-map \( f \) of \( \mathbb{P}_2 \) that leaves \( Q \) invariant, \( m_Q(f) \) denotes the multiplier of its restriction to \( Q \). Given \( 0 \neq m \in \mathbb{Z}(C) \), we wish to construct self-maps of \( \mathbb{P}_2 \) that leave \( Q \) invariant, with \( m_Q(f) = m \). To do this, we must find \( n \in \mathbb{C} \) so that the self-map \( [\cdot] \to [mt - n] \) of \( C \) induces through \( \nu \) a regular self-map \( g \) of \( Q \) with \( g^* \mathcal{O}_Q(1) \cong \mathcal{O}_Q(|m|^2) \).

Definition 2.6. Given \( 0 \neq m \in \mathbb{Z}(C) \), \( R_Q(m) \) denotes the set of points \( [n] \in C \) with the property that the self-map \( [\cdot] \to [mt - n] \) of \( C \) induces through \( \nu \) a self-map of \( Q \) that admits rational extensions to \( \mathbb{P}_2 \). Let \( r_Q(m) \) be the cardinality of \( R_Q(m) \).

With a choice of normalization \( \nu \) and group structure \( [\cdot] \), \( R_Q(m) \) is identified with the set of self-map of \( Q \) with multiplier \( m \) that extend rationally to \( \mathbb{P}_2 \).


3.1. Multiplicities. Denote by \( m_a(A) \) the multiplicity, and by \( T_a(A) \) the tangent of an irreducible curve germ \( (A, a) \subset (C^2, a) \).

Lemma 3.1. Let \( (A, a) \subset (C^2, a) \) and \( (B, b) \subset (C^2, b) \) be irreducible curve germs, with normalizations \( (\hat{A}, \hat{a}) \xrightarrow{\nu_A} (A, a) \) and \( (\hat{B}, \hat{b}) \xrightarrow{\nu_B} (B, b) \). Let \( (C^2, a) \xrightarrow{f} (C^2, b) \) be a finite map germ with \( f(A, a) = (B, b) \). Denote by \( (A, a) \xrightarrow{g} (B, b) \) the restriction of \( f \), and by \( (\hat{A}, \hat{a}) \xrightarrow{\hat{g}} (\hat{B}, \hat{b}) \) the lifting of \( g \) through \( \nu_A \) and \( \nu_B \).

1. If \( d\hat{g}(\hat{a}) \neq 0 \) then \( m_a(A) \leq m_b(B) \), and \( m_a(A) = m_b(B) \) iff \( df(a)_{|T_a(A)} \neq 0 \).
2. If \( \operatorname{d}g(\bar{a}) = 0 \) and \( m_a(A) \leq m_b(B) \), then \( \operatorname{df}(a)|_{T_a(A)} = 0 \).

**Proof.** We explicite the Puiseux series of \((A, a)\) and \((B, b)\), and then identify the coefficients in the Taylor series at \( \bar{a} \) of \( \nu_B \circ \nu_A \).

Choose local coordinates \((x, y)\) near \( a = (0, 0) \in \mathbb{C}^2 \), \((u, v)\) near \( b = (0, 0) \in \mathbb{C}^2 \), \( s \) near \( \bar{a} = 0 \in \mathbb{A} \), and \( t \) near \( \bar{b} = 0 \in \mathbb{B} \) so that \( T_a(A) = (y = 0), T_b(B) = (v = 0), \nu_A(s) = (s^n, O_{m+1}(s)), \) and \( \nu_B(t) = (t^p, O_{p+1}(t)) \), with \( m_a(A) = p = m_b(B) \).

Here, \( O_{m+1}(s) \) denotes a holomorphic function involving terms of degree at least \( m \) or \( 1 \). Write \( f(x, y) = (u, v) = (ax + by + O_2(x, y), gx + dy + O_2(x, y)) \), and \( \bar{g}(s) = t = ks + O_2(s) \). Calculate \( \nu_B \circ \nu_A(s) = (k^{p-s} + O_{p+1}(s), O_{p+1}(s)) \), and \( f(v) = (as^n + O_{m+1}(s), s^{m+1} + O_{m+1}(s)) \). Therefore, \( \alpha = m = m_b(B) = m_a(A) = p = m_b(B) \).

Assume first that \( \operatorname{d}g(\bar{a}) \neq 0 \), i.e. \( k \neq 0 \). Then \( p \geq m \), or else \( k^{p-s} = O_{p+1}(s) \). If \( p = m \), then \( \alpha = k^m \) and \( \gamma = 0 \), hence \( \operatorname{df}(x)(\partial_x) = \alpha \partial_x \). If \( p > m \), then \( \alpha = \gamma = 0 \), hence \( \operatorname{df}(0)(\partial_x) = 0 \).

If \( k = 0 \) and \( m \leq p \), then \( \alpha = \gamma = 0 \), hence \( \operatorname{df}(0)(\partial_x) = 0 \).

**Lemma 3.2.** With the notations of Lemma 3.1, assume that \((A, a) = (B, b)\).

1. If \( \operatorname{d}g(\bar{a}) \) acts on \( T_a(\mathbb{A}) \) as multiplication by \( k \in \mathbb{C} \), then \( \operatorname{df}(a)(\partial_x) = k \times \operatorname{df}(a)(\partial_x) \).

2. If \( f^*(A, a) > (A, a) \) and \( m_a(A) > 1 \), then \( \operatorname{d}g(\bar{a}) = 0 \) and \( \operatorname{df}^2(a) = 0 \).

**Proof.** The first statement follows from the proof of Lemma 3.1. To show the second statement, choose coordinates \((x, y)\) near \( a \in \mathbb{C}^2 \), and \( s \) near \( \bar{a} = 0 \in \mathbb{A} \), so that \( \nu_A(s) = (s^n, s^m + O_{m+1}(s)), \) with \( 1 \leq m < n \), and \( n/m \notin \mathbb{Z} \). Write \( \bar{g}(s) = ks + O_2(s) \), and \( f(x, y) = (u, v) \), with \( u = ax + by + O_2(x, y) \), and \( v = gx + dy + O_2(x, y) \).

Since \( (A, 0) \) is defined by \( y + O_m(x, y), f^*(A, 0) \) is defined by \( v + O_{m+1}(u, v) \). Since \( f^*(A, 0) > (A, 0) \), we get \( v = d^2v + O_{m+1}(u, v) = (y + O_m(x, y)) \). Therefore, \( v = O_m(x, y) \), i.e. \( v = \delta = 0 \).

Since \( \nu_A \circ \nu_B(s) = (k^{n-s} + O_{n+1}(s), k^{n-s} + O_{n+1}(s)), \) and \( f^*(A, 0) = (u^{s^n} + O_{m+1}(s), O_{m+1}(s)) \), we get \( v^{s^n} = k^{n-s} + O_{m+1}(s) \). Assume \( k \neq 0 \), and write \( v(x, y) = w(x) + O(O_{m+1}(x, y)) \). Then \( w^{s^n} = k^{n-s} + O_{m+1}(s) \), hence \( w^{s^n} = k^{n-s} + O_{m+1}(s) \). Clearly, \( w \neq 0 \). Write \( w(x) = h x^p + O_{p+1}(x) \), with \( h \neq 0 \) and \( p > 2 \). If \( m > n \), then \( h = 0 \). If \( m > n \), then \( n/m \notin \mathbb{Z} \) if \( m > n \), then \( k = 0 \).

By (1), \( \alpha = k^m \), hence \( \alpha = 0 \). It follows that \( \operatorname{df}^2(a) = (\operatorname{df}(a))^2 = 0 \).

**Lemma 3.3.** Let \((Q, 0) \subset (\mathbb{C}^2, 0)\) be a curve germ with branches \((A_i, 0)\) normalized by \((\mathbb{A}_i, \mathbb{A}_i) \rightarrow (A_i, 0) \). Let \((\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0) \) be a finite map germ that satisfies \( f(Q, 0) \subset (Q, 0) \), inducing \((A_i, 0) \rightarrow (A_i, 0) \). Let \((\mathbb{A}_i, \mathbb{A}_i) \rightarrow (A_i, 0) \) be the lifting of \( g_i \) through \( f \), for all \( i \). Let \( \operatorname{d}g_i(\bar{a}_i) \neq 0 \) for all \( i \), then \( \operatorname{df}(0)|_{T_a(A_i)} \neq 0 \) for all \( i \).

**Proof.** Since every \((A_i, 0)\) is pre-periodic for \( f \), replacing \( f \) by an iterate, we may assume that \( f^2(A_i) = f(A_i) \) for all \( i \).

When \( f(A_i) \neq A_i \), we have \( m_0(f(A_i)) = 1 = m_0(A_i) \) and \( \operatorname{df}(0)|_{T_a(A_i)} \neq 0 \). Indeed, in this case \( f^*(f(A_i)) \geq f(A_i) + A_i > f(A_i) \), hence, by Lemma 3.2.2, \( m_0(f(A_i)) = 1 \). By Lemma 3.1.1, \( m_0(A_i) \leq m_0(f(A_i)) \), hence \( m_0(A_i) = 1 \), and then \( \operatorname{df}(0)|_{T_a(A_i)} \neq 0 \).

When \( f(A_i) = A_i \), Lemma 3.1.1 implies that \( \operatorname{df}(0)|_{T_a(A_i)} \neq 0 \).
3.2. **Torsion.** Let \( \mathbb{C} \xrightarrow{[1]} (C,+,0) \) be a smooth elliptic curve. Given a divisor \( D \) on \( C \), let \( s(D) := \sum_{c \in D} c \) (multiplicities are counted in all such sums and products indexed by divisors). By Jacobi's theorem, \( D \simeq 0 \) iff \( \deg(D) = 0 \) and \( s(D) = 0 \).

Given \( 0 \neq m \in \mathbb{Z}(C) \), let \( C_m \) be the kernel of the morphism \( [t] \mapsto [mt] \), and \( c_m := s(C_m) \). Note that \( c_m \in C_m \cap C_2 \). When \( m \in \mathbb{Z} \) or \( |m|^2 \) is odd, \( c_m = 0 \).

**Remark 3.4.** Let \( D \) be a divisor on \( C \) with \( 0 \neq \epsilon := \deg(D) \) and \( s(D) = 0 \). Given \( 0 \neq m \in \mathbb{Z}(C) \) and \( n \in \mathbb{C} \), the map \( C \xrightarrow{\cdot m} C, g[t] = [mt - n] \), satisfies \( g^*D \simeq |m|^2D \) if \( e([mn]) = ec_m \). In this case, \( g(C_{[m]^2}) \subset C_{[m]} \).

**Proof.** We calculate \( s(g^*D) = e([mn] + c_m) \), hence \( s(g^*D) = 0 \) if \( e([mn]) = ec_m \). Such \( g \) satisfies \( e([m]^2[n]) = 0 \), hence \( g(C_{[m]^2}) \subset C_{[m]} \).

Given a divisor \( D \) on \( C \), write \( D \simeq r \) if there exists a positive integer \( k \) so that \( kD \simeq 0 \). Let \( r(D) \) be the smallest such \( k \). When \( D \neq 0 \), let \( r(D) := +\infty \). Given \( a, b \in C \), write \( a \simeq r b \) if \( (a) - (b) \simeq 0 \). Let \( r(a,b) := r((a) - (b)) \).

**Remark 3.5.** Let \( C \xrightarrow{h} C \) be a self-map of a smooth elliptic curve, \( 0 \neq m = m(h) \). Given any points \( a \) and \( b \) in \( C \), \( \frac{\tau(a,b)}{|m|^2} \leq \tau(h(a),h(b)) \leq \tau(a,b) \).

If \( |m| > 1 \) and \( a \) is pre-periodic for \( h \), then \( a \simeq r b \) if \( b \) is pre-periodic for \( h \).

**Proof.** We may assume that \( h \) is a morphism of \( (C,+,0) \). If \( k(a - b) = 0 \), then \( k(h(a) - h(b)) = k(0) = 0 \). If \( k(h(a) - h(b)) = 0 \), then \( k|m|^2(a - b) = 0 \).

Let \( a \) be pre-periodic for \( h \). Replacing \( h \) by an iterate, we may assume that \( h^2(a) = h(a) \), hence \( |m|^2 - m^2a = 0 \). When \( |m| > 1 \), we get \( a \simeq r 0 \).

If \( b \simeq r a \), then \( b \simeq r 0 \), i.e., \( kb = 0 \) for some \( 0 < k \in \mathbb{Z} \). The \( h \)-orbit of \( b \) is contained in the finite set of points \( c \in C \) with \( kc = 0 \), hence \( b \) is pre-periodic.

Most elliptic plane curves do not admit self-maps of degree greater than 1:

**Proposition 3.6.** Let \( Q \) be an elliptic plane curve with normalization \( C \xrightarrow{\nu} Q \). Given \( Q \xrightarrow{g} Q \) with \( \deg(g) > 1 \), let \( C \xrightarrow{h} C \) be the lifting of \( g \) through \( \nu \). Then:

1. The singular branches of \( Q \) are pre-periodic for \( h \). If \( a \) and \( b \) are singular branches of \( Q \), then \( a \simeq r b \).

2. If \( a \) and \( b \) are branches of \( Q \) with \( \nu(a) = \nu(b) \), then \( a \simeq r b \).

**Proof.** Let \( S \subset C \) be the finite set of singular branches of \( Q \). By Lemma 3.1.(1), \( h(S) \subset S \). Remark 3.5 finishes the proof of (1).

To prove (2), let \( q = \nu(a) = \nu(b) \). If \( q \) is pre-periodic for \( g \), then \( a \) and \( b \) are pre-periodic for \( h \), hence \( a \simeq r b \). If \( q \) is not pre-periodic for \( g \), replacing \( g \) by an iterate we may assume that \( (Q,g(q)) \) is irreducible. Then \( h(a) = h(b) \), hence \( a \simeq r b \).

3.3. **Ordinary singularities.** Let \( Q \) be an elliptic plane curve, with normalization \( C \xrightarrow{\nu} Q \) and group structure \( \mathbb{C} \xrightarrow{[1]} C \). A germ \( (Q,q) \) is an ordinary singularity iff \( m_q(Q) = 2 \) and the proper transform of \( (Q,q) \) through the blow-up of \( \mathbb{P}^2 \) at \( q \) is smooth. In suitable local coordinates near \( q \), an ordinary singularity \( (Q,q) \) is either the cusp \( (y^2 = x^3) \) or the node \( (xy = 0) \).

**Lemma 3.7.** Given \( C \xrightarrow{\phi} \mathbb{P}^1 \), let \( \psi := \phi^{-1} \) be defined on the smooth locus of \( Q \). If \( \psi \) is regular on \( Q \), the following conditions are satisfied:

1. If \( [a] \) is a singular branch of \( Q \), then \([a]\) is critical for \( \phi \).
2. If $v[a] = v[b]$, then $\phi[a] = \phi[b]$.

When $Q$ has ordinary singularities, these conditions imply that $\psi$ is regular on $Q$.

Proof. Assume $\psi$ is regular on $Q$, so that $\psi v = \psi$. If $v[a] = v[b]$, clearly $\phi[a] = \phi[b]$. The singular branches of $Q$ are critical for $v$, hence they are critical for $\phi$.

Assume that $C$ has ordinary singularities. Given a node $v[a] = v[b]$, let $\psi_a, \psi_b$ be the restrictions of $\psi$ to the branches $[a], [b]$ (respectively). Clearly, $\psi_a$ and $\psi_b$ are regular. If $\phi[a] = \phi[b]$, then $\psi_a(v[a]) = \psi_b(v[a])$, hence $\psi$ is regular at $v[a]$. Given a cusp $v[a]$, in suitable local coordinates $t$ near $[a]$ and $(x, y)$ near $v[a]$, $v(t) = (t^2, t^3)$. If $[a]$ is critical for $\phi$, then $\phi(t)$ is a holomorphic function of $t^2$ and $t^3$.

\[ \Box \]

Lemma 3.8. Given $C \xrightarrow{h} C$, let $g := vhv^{-1}$ be defined on the smooth locus of $Q$.

If $g$ is regular on $Q$, the following conditions are satisfied:
1. If $[a]$ is a singular branch of $Q$, then $h[a]$ is also singular.
2. If $v[a] = v[b]$, then $vh[a] = vh[b]$.

When $Q$ has ordinary singularities, these conditions imply that $g$ is regular on $Q$.

Proof. Assume $g$ is regular on $Q$, so that $gv = vh$. If $v[a] = v[b]$, clearly $vh[a] = vh[b]$. If $[a]$ is a singular branch, Lemma 3.1 (1) implies that $h[a]$ is also singular.

Assume that $C$ has ordinary singularities. Given a node $v[a] = v[b]$, let $g_a, g_b$ be the restrictions of $g$ to the branches $[a], [b]$ (respectively). Clearly, $g_a$ and $g_b$ are regular. If $vh[a] = vh[b]$, then $g_a(v[a]) = g_b(v[a])$, hence $g$ is regular at $v[a]$. If $h(v[a])$ is a cusp, choose, as before, local coordinates $t$ near $[a]$ and $(x, y)$ near $v[a]$, so that $v(t) = (t^2, t^3)$. Since $[a]$ is critical for $vh$, $vh(t)$ is a holomorphic function of $t^2$ and $t^3$, hence $g$ is regular at $v[a]$.

3.4. Invariants at singular points. Recall the definition and basic properties of the Weierstrass functions $\sigma, \zeta$ and $P$. Let $\Omega$ be a lattice in $\mathbb{C}$, with associated elliptic curve $\mathbb{C} \xrightarrow{1} \mathbb{C}/\Omega = (C, +, 0)$. Fix a positively oriented basis $(\lambda_1, \lambda_2)$ in $\Omega$.

Given two lattice points $\omega_t = a_1\lambda_1 + b_1\lambda_2 \in \Omega$, $\det(\omega_1, \omega_2) := \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

The odd entire function $\sigma(t) := t \prod_{0 \neq \omega \in \Omega} (1 - \frac{t}{\omega}) \exp(\frac{1}{2} - \frac{t^2}{2\omega^2})$ has the translation property $\frac{\sigma(t + \omega)}{\sigma(t)} = e(\omega) \exp\left(\frac{1}{2} \eta(\omega) t \right) \exp(\eta(\omega)t)$, for all $t \in \mathbb{C}$ and $\omega \in \Omega$. Here, $e(\omega) = \begin{cases} 1, & \text{when } \omega \in 2\Omega \\ -1, & \text{when } \omega \notin 2\Omega \end{cases}$, and $\Omega \xrightarrow{\eta} \mathbb{C}$ is a group morphism that satisfies the Legendre relation $\begin{vmatrix} \eta(\omega_1) & \omega_1 \\ \eta(\omega_2) & \omega_2 \end{vmatrix} = 2\pi i \det(\omega_1, \omega_2)$.

The odd meromorphic function $\zeta := (\log \sigma)' = \frac{1}{\Omega} + \sum_{0 \neq \omega \in \Omega} \left( \frac{1}{\omega} - \frac{1}{\omega - \frac{1}{\omega}} \right)$ has the translation property $\zeta(t + \omega) = \zeta(t) + \eta(\omega)$, for all $t \in \mathbb{C}$ and $\omega \in \Omega$.

The function $P := \zeta' = \frac{-1}{\Omega} - \sum_{0 \neq \omega \in \Omega} \left( \frac{1}{(\omega - \frac{1}{\omega})^2} - \frac{1}{\omega} \right)$ is even and $\Omega$-periodic.

Given a divisor $A$ on $\mathbb{C}$, let $[A] := \sum_{a \in A} \{a\}$, and $s(A) := \sum_{a \in A} a$. When $[A] \simeq 0$, the meromorphic function $\Phi_A(t) := \exp(\eta(s(A)t)) \prod_{a \in A} \sigma(t - a)$ is $\Omega$-periodic, inducing an elliptic function $C \xrightarrow{\phi_A} \mathbb{P}^1$, with principal divisor $([A]$).
$C \xrightarrow{h} C$, \( h[l] := [mt - n] \), and the meromorphic function \( C \xrightarrow{\Phi} \mathbb{P}^1 \), \( \Phi(t) := \exp(t\eta(\text{emn})) \prod_{a \in \mathcal{D}} \frac{\sigma^{m(t - a)}}{\sigma^m} \). Then \( \Phi \) is \( \Omega \)-periodic, inducing on \( C \) an elliptic function \( C \xrightarrow{\phi} \mathbb{P}^1 \) with \( \phi = h^* D - m^2 D \).

**Proof.** Let \( \lambda = \text{emn} \), and fix \( \omega \in \Omega \). Since \( m \in \mathbb{Z} \) and \( s(\mathcal{D}) = 0 \), the translation property of \( \sigma \) and the Legendre identity imply that

\[
\frac{\Phi(t + \omega)}{\Phi(t)} = \exp(\omega(\eta(\lambda))) \prod_{a \in \mathcal{D}} \frac{e(m^2)}{e(m)} \exp\left(\frac{m(\omega - \lambda)}{\sigma^m}\right) = \exp(\omega(\lambda) - \lambda(\omega)) \exp((m^2 - m)s(\mathcal{D})\eta(\lambda)) = 1.
\]

Let \( Q \) be an elliptic plane curve of degree \( e \). Choose a normalization \( C \xrightarrow{\nu} Q \), and a group structure \( C \xrightarrow{\iota} C/\Omega = (C, +, 0) \) so that \( s(\iota \circ Q(1)) = 0 \). Given a line \( l \) in \( \mathbb{P}^2 \), let \( l_C := (\iota \nu)^* l \), where \( Q \xrightarrow{\iota} \mathbb{P}^2 \) is the inclusion map. Given a divisor \( D \) on \( C \) with \( s(D) = 0 \), \( \tilde{D} \) denotes a divisor on \( C \) with \( \tilde{D} = D \) and \( s(\tilde{D}) = 0 \).

**Definition 3.10.** Given two points \( a, b \) on \( C \) and a line \( l \) in \( \mathbb{P}^2 \) not passing through \( \nu[a] \) or \( \nu[b] \), define

\[
\alpha(a, l) := \sum_{t \in \mathcal{C}} \zeta(a - t) \quad \text{and} \quad \alpha(a, b, l) := \prod_{t \in \mathcal{C}} \frac{\sigma(a - t)}{\sigma(b - t)}.
\]

The translation properties of \( \zeta \) and \( \sigma \) imply that \( \alpha(a, l) \) and \( \alpha(a, b, l) \) do not depend on the choice of divisor \( \tilde{D} \) on \( C \) satisfying \( [\tilde{D}] = l \) and \( s(\tilde{D}) = 0 \).

**Remark 3.11.** (3.4) The translation properties of \( \alpha \) come from those of \( \zeta \) and \( \sigma \). Given \( \omega, \lambda \in \Omega \), \( \alpha(a + \omega, l) = \alpha(a, l) = \alpha(a, b, l) = \exp(\eta(\lambda)(a - b)) \),

and \( \frac{\alpha(a, b + \omega, l)}{\alpha(a, b, l)} = \left( \frac{\alpha(a, b + \omega, l)}{\alpha(a, b, l)} \right)^\omega \exp\left( \frac{\omega}{\sigma^m}(\eta(\omega) - \eta(\lambda)\lambda) \right) \exp(e(\eta(\omega)\alpha - \eta(\lambda)\beta)) \).

Recall that \( \mathbb{P}^2 \) denotes the space of lines in \( \mathbb{P}^2 \). Given \( p \neq q \) in \( \mathbb{P}^2 \), \( L(p, q) \) is the line passing through \( p \) and \( q \). Given \( p \in \mathbb{P}^2 \), \( \bar{p} := \{ l \in \mathbb{P}^2 : p \in l \} \).

**Proposition 3.12.** When \( \nu[a] = \nu[b] \), \( \alpha(a, b, l) \) is independent of \( l \in \mathbb{P}^2 \setminus \nu[a] \).

When \( [a] \) is a singular branch of \( Q \), \( \alpha(a, l) \) is independent of \( l \in \mathbb{P}^2 \setminus \nu[a] \).

**Proof.** Fix two lines \( l_0 \neq l_\infty \) in \( \mathbb{P}^2 \) not passing through \( q := \nu[a] \), and let \( p := l_0 \cap l_\infty \), \( l_1 := L(p, q) \). Let \( C \xrightarrow{\phi} \mathbb{P}^1 \) be the central projection \( \phi[x] := L(p, [x]) \). Given \( l \cap p \), a local study near the intersection of \( Q \) with \( l \) shows that \( \phi^*[l] = l_C \). Pick coordinates in \( \mathbb{P}^1 \) with \( l_0 = 0 \) and \( l_\infty = \infty \). Let \( A := (l_0)_C - (l_\infty)_C \). Clearly, \( \deg(A) = 0 \) and \( s(A) = 0 \). Since \( (\phi_A) = (l_0)_C - (l_\infty)_C \), there exists \( 0 \neq k \in \mathbb{C} \) with \( \phi = k\phi_A \).

If \( \nu[b] = q \), then \( \phi[a] = \phi[b] = l_1 \), hence \( \frac{\alpha(a, b, l_0)}{\alpha(a, b, l_\infty)} = \frac{\phi(a)}{\phi(b)} = \frac{\phi}[\phi] = 1 \).

If \( [a] \) is a singular local branch of \( Q \), then \( \phi^*[l_1] = (l_1)_C \), \( \nu^*[l_1] = \nu^*[q] \geq 2[a] \), i.e. \( \phi'[a] = 0 \). We get \( \alpha(a, l_0) - \alpha(a, l_\infty) = \frac{\phi'(a)}{\phi} = \phi'[a] = 0 \).

**Definition 3.13.** Let \( 0 \neq m \in \mathbb{Z} \) and \( n \in C_{em} \).

Given a singular branch \( [a] \) of \( Q \) so that \([ma - n]\) is also singular, define

\[
\alpha_{m,n}(a) := ma(ma - n) - m^2 \alpha(a) + \eta(\text{emn}).
\]
Given two branches \([a], [b]\) of \(Q\) with \(\nu[a] = \nu[b]\) and \(\nu[ma - n] = \nu[mb - n]\), define
\[
\alpha_{m,n}(a, b) := \frac{\alpha(ma - n, mb - n)}{\alpha_m(a, b)} \cdot \exp((a - b)\eta(emm)).
\]

Clearly, \(\alpha_{m,n}(a)\) and \(\alpha_{m,n}(a, b)\) are well-defined, and depend only on \([a], [b], [n]\). Moreover, \(\alpha_{m,n}(a, b)\alpha_{m,n}(b, a) = 1\).

**Theorem 3.14.** Let \(Q\) be an elliptic plane curve of degree \(e\). Choose a normalization \(C \xrightarrow{\phi} Q\), and a group structure \(C \xrightarrow{\phi} \mathbb{C}/\Omega = (C, +, 0)\) so that \(s(\nu^*\Omega_Q(1)) = 0\). Assume that \(Q\) has ordinary singularities. Then, given \(0 \neq m \in \mathbb{Z}\), \(R_Q(m)\) is formed by the points \([n] \in C_{cm}\) with the following properties:

1. If \([a]\) is a singular branch of \(Q\), then \([ma - n]\) is singular and \(\alpha_{m,[n]}[a] = 0\).
2. If \(\nu[a] = \nu[b]\), then \(\nu[ma - n] = \nu[mb - n]\) and \(\alpha_{m,[n]}([a], [b]) = 1\).

**Proof.** Given \(0 \neq m \in \mathbb{Z}\) and \([n] \in C\), let \(h[t] := [mt - n]\), and define \(g := \nu h \nu^{-1}\) on the smooth locus of \(Q\). Fix a generic line \(l\) in \(\mathbb{P}^2\), and let \(l_Q := i^*l, l_C := \nu^*l_Q\).

If \([n] \in R_Q(m)\), \(g\) extends rationally to \(\mathbb{P}^2\). By Remark 2.4, \(g^*l_Q \simeq m^2l_Q\). Since \(g\) is a singular then \(h[a]\) is singular. If \(\nu[a] = \nu[b]\), then \(\nu h[a] = \nu h[b]\). Now, there exists \(Q \xrightarrow{\psi} \mathbb{P}^1\) with \((\psi) = g^*l_Q - m^2l_Q\), hence \((\psi) = h^*l_C - m^2l_C\). By Lemma 3.9 applied to \(l_C\), we may assume that \(\psi\) is induced by the \(\Omega\)-periodic function \(\Phi(t) := \exp(\eta t\exp[/\mathrm{en}])\) \(\prod_{\alpha \in L_{C}} \frac{\alpha^{m, t - n - \alpha}}{\sigma(t - \mathrm{n})}\). By Lemma 3.7, if \([a]\) is a singular, then

\[
\Phi(a) = 0, \text{ i.e. } \alpha_{m,[n]}[a] = 0. \text{ If } \nu[a] = \nu[b], \text{ then } \Phi(a) = \Phi(b), \text{ i.e. } \alpha_{m,[n]}([a], [b]) = 1.
\]

Assume that \([n] \in C_{cm}\) satisfies the two conditions. Since the singularities of \(Q\) are ordinary, \(g\) is regular, by Lemma 3.8. By Remark 3.4, \(h^*l_C \simeq m^2l_C\). Let \(C \xrightarrow{\phi} \mathbb{P}^1\) be induced by \(\Phi\), with \(\Phi\) defined as above. By Lemma 3.7, \(\psi := \phi \nu^{-1}\) is regular on \(Q\). Since \((\phi) = h^*l_C - m^2l_C\), we get \((\psi) = g^*l_Q - m^2l_Q\), hence \(g^*l_Q \simeq m^2l_Q\). Remark 2.4 finishes the proof. \(\square\)

**Remark 3.15.** Given any elliptic plane curve \(Q\), \(R_Q(m) \subset C_{cm}\), and the points of \(R_Q(m)\) satisfy the properties stated in Theorem 3.14.

As an application of Theorem 3.14, we have the following.

**Corollary 3.16.** With the notations of Theorem 3.14, assume that \(Q\) is singular, and that the singularities of \(Q\) are ordinary cusps. If \(0 \neq k \in \mathbb{Z}\) has the property that \(k[a] = 0\) for all cusps \([a]\), then, given \(m \in \mathbb{Z}\) with \(|m| > 1\), \(R_Q(m) \subset R_Q(m + k)\).

**Proof.** Fix \([n] \in R_Q(m)\). Given a cusp \([a_0]\), let \(a_{j+1} := ma_j - n, j \geq 0\). Since \([a_j]\) is a cusp for all \(j, [a_s] = [a_{s+1}]\) for some \(0 \leq s \leq r\). Clearly, \(a_j = ma_j - n\).

By Remark 3.5, there exists such \(k\). Let \(\omega_j := ka_j, \omega := k\), and \(\alpha_j := \frac{k}{e}(\alpha(a_j))\).

Since \(\alpha_{m,n}[a_j] = 0\), we get \(a_{j+1} = ma_j - \eta(\omega).\) By Remark 3.4, \(k\eta(a_r - a_s) = \alpha_r - \alpha_s = (m^r - m^s)(\alpha_0 - \frac{m\omega_0}{m-1})\), hence \(a_0 = \eta(\omega_0)\).

We calculate \(\frac{k}{e(m+k)} \alpha_{m,k,n}(a_0) = \frac{k}{e}(a_1 + \omega_0) - (m+k)\alpha_0 + \eta(\omega) = \alpha_1 + k\eta(\omega_0) - (m + k)\eta(\omega) + \eta(\omega) = 0\). \(\square\)

4. **Invariant Critical Components.**

In this section we prove that, given an elliptic plane curve \(Q\), there do not exist self-maps of \(\mathbb{P}^2\) for which \(Q\) is critical and invariant.
Theorem 4.1. Given a plane curve $Q$, the following are equivalent:

1. There exist self-maps of $\mathbb{P}^2$ for which $Q$ is invariant and critical.
2. The curve $Q$ is rational.

Proof. Assume that $Q$ is invariant and critical for a rational self-map $f$ of $\mathbb{P}^2$. By Remark 2.1, $Q$ is rational or elliptic. Assume that $Q$ is elliptic. Fix a normalization map $C \rightarrow Q$, and consider the incidence surface $S := \{(c, l) \in C \times \mathbb{P}^2 : \nu(c) = l\}$, with canonical projections $S \rightarrow C$, $p(c, l) = c$, and $S \rightarrow \mathbb{P}^2$, $\pi(c, l) = l$. Clearly, $S$ is smooth, $S \rightarrow C$ is a ruled surface, and $\pi$ is finite, with $\deg(\pi) = \deg(Q) = e$. Fix $r \in \mathbb{P}^2$, and let $L$ be the graph of the projection $C \ni c \mapsto L(r, \nu(c)) \in \mathbb{P}^2$. For generic $r$, we have $\pi^*r = L$, hence $L^2 = e$.

By Lemma 3.1 and Lemma 3.2, $Q$ cannot have singular branches, hence the curve $Q$ dual to $Q$ has degree $2e$ in $\mathbb{P}^2$. Let $T$ be the graph of the map dual to $\nu$, $C \ni c \mapsto T_{\nu(c)}c \in \mathbb{P}^2$. We see that $LT = \deg(Q) = 2e$.

Given $q \in Q$, the differential $df(q)$ has 1-dimensional kernel $X_q$, which we identify with the line in $\mathbb{P}^2$ whose tangent space at $q$ is $X_q$. Let $X$ be the graph of the map $C \ni c \mapsto X_{\nu(c)} \in \mathbb{P}^2$. Clearly, $XT = 0$.

Now, $L$, $T$, and $X$ are sections in the ruled surface $S \rightarrow C$. Let $H$ be a minimal section of $S$, with $H^2 = n$, and denote by $F$ the class of the fibers of $p$, modulo the numerical equivalence $\sim$ of divisors on $S$. For some non-negative integers $l$, $t$, $x$, we have $L \sim H + lF$, $T \sim H + tF$ and $X \sim H + xF$, with $n + 2l = e$, $n + l + t = 2e$ and $n + x + t = 0$. We get $n = e - 2l$, $t = l + e$ and $x = l - 2e$. Since $t > 0$, we have $T \neq H$, hence $-x = n + t = TH \geq 0$. Therefore, $x = 0$, $l = 2e$ and $n = -2e$. Since $l > 0$, we have $L \neq H$, hence $LH \geq 0$. But $LH = -e < 0$, a contradiction.

Vice versa, assume that $Q$ is rational of degree $e$, with normalization $\mathbb{P}^1 \rightarrow Q$.

Fix a rational map $\mathbb{P}^2 \rightarrow \mathbb{P}^1$ of algebraic degree $d \geq 2$, that satisfies $I(\phi) \cap Q = \emptyset$. The rational self-map $f := \nu \phi$ of $\mathbb{P}^2$ is degenerate with image $Q$, has $d(f) = de \geq 2e$, and satisfies $I(f) \cap Q = \emptyset$. Adding to the components of $f$ generic multiples of $h^2$, where $h$ is the homogeneous equation of $Q$, we obtain regular self-maps of $\mathbb{P}^2$ for which $Q$ is invariant and critical.

5. Julia set.

A domain $D$ in a complex manifold $M$ is hyperbolically embedded in $M$ iff for any two sequences $(x_n)_n$, $(y_n)_n$ of points in $D$, if $x_n \rightarrow x \in M$, $y_n \rightarrow y \in M$, and $d_D(x_n, y_n) \rightarrow 0$, then $x = y$. Here, $d_D$ denotes the Kobayashi pseudo-distance.

Given two complex spaces $X$ and $Y$, let $\text{Hol}(X, Y)$ denote the space of holomorphic maps from $X$ to $Y$, endowed with the compact-open topology. In [9] it is shown that if $D \subset M$ is relatively compact and hyperbolically embedded, then, given any complex manifold $U$, $\text{Hol}(U, D)$ is relatively compact in $\text{Hol}(U, M)$.

An irreducible complex space $X$ is Brody hyperbolic iff it admits no non-constant holomorphic maps $\mathbb{C} \rightarrow X$. We will use Green’s results on the hyperbolicity of the complement of a hypersurface.

Theorem 5.1 ([7]). Let $M$ be a compact complex manifold and $H \subset M$ a hypersurface, with irreducible components $H_i$, $1 \leq i \leq m$. Assume that:

1. $M \setminus H$ is Brody hyperbolic;
2. $\bigcap_{i \in I} H_i \setminus \bigcup_{j \in J} H_j$ is Brody hyperbolic for every partition $I \sqcup J = \{1, \ldots, m\}$.


Then $M \setminus H$ is completely hyperbolic, and hyperbolically embedded in $M$.

**Theorem 5.2** ([6], [16]). Let $M$ be a projective manifold and $\mathbb{C}^4 \xrightarrow{f} M$ a holomorphic map that omits at least $\dim(M) + 2$ ample hypersurfaces in $M$. Then the image of $f$ is contained in some hypersurface of $M$.

A quasi-projective irreducible curve $Q_0$ is hyperbolic iff its normalization $\tilde{Q}_0$ is a hyperbolic Riemann surface. $Q_0$ is hyperbolic iff there are no nonconstant maps from $\mathbb{C}$ to $Q_0$. We will use the following result.

**Theorem 5.3** ([1]). Let $C$ be a plane curve with at least four irreducible components, at least one of them irrational. Then there are at most finitely many irreducible curves $B \subset \mathbb{P}^2$ with the property that $B \setminus C$ is not hyperbolic.

Given a self-map $\mathbb{P}^n \xrightarrow{f} \mathbb{P}^n$, $J(f)$ denotes its Julia set. When $n = 1$, all but at most two points have the property that $J(f)$ is contained in the closure of their backward orbit. When $n = 2$, it may happen that no point has this property. (For example, when $f$ has a chaotic completely invariant line.)

**Theorem 5.4.** If an elliptic plane curve $Q$ is invariant for a map $\mathbb{P}^2 \xrightarrow{J} \mathbb{P}^2$ with $d(f) > 1$, then $J(f)$ equals the closure of the backward $f$-orbit of any point on $Q$.

**Proof.** For $n \geq 0$, let $Q_n := f^{-n}(Q)$. Denote by $\mathcal{J}$ the closure of $\bigcup_{n \geq 0} Q_n$. Since $Q$ is elliptic, $\mathcal{J}$ is the closure of the backward orbit of any point on $Q$, and $\mathcal{J} \subset J(f)$. We need to show that the sequence of iterates of $f$ is normal on $\Omega = \mathbb{P}^2 \setminus \mathcal{J}$. It suffices to find some $n_0$ such that $\mathbb{P}^2 \setminus Q_{n_0}$ is hyperbolically embedded in $\mathbb{P}^2$.

The irreducible components of $Q_n$ are mapped by $f^n$ to $Q$, hence they are irrational. Note that $Q_n$ has at least $n + 1$ irreducible components. This can easily be seen by induction, since no irrational plane curve is completely $f$-invariant ([5]).

By Theorem 5.3, we deduce that when $n \geq 3$ there are at most finitely many irreducible curves $B \subset \mathbb{P}^2$ with the property that $B \setminus Q_n$ is not hyperbolic.

We show that for every irreducible curve $B \subset \mathbb{P}^2$ there exists some positive integer $n$ so that $B \setminus Q_n$ is hyperbolic. Assume this is not true, and let $B$ denote the finite set of irreducible curves $B \subset \mathbb{P}^2$ with the property that $B \setminus Q_n$ is not hyperbolic for all $n \geq 0$. Since $f(B \setminus Q_{n+1}) \subset f(B \setminus Q_n)$, $f$ acts on $B$. Pick an $f$-periodic curve $B \in B$. Replacing $f$ by an iterate, we may assume that $f(B) = B$. Let $B \xrightarrow{\nu} B$ denote the map induced by $f$. Note that $B$ is rational, since any Zariski open subset of an irrational curve is hyperbolic. Let $\mathbb{P}^1 \xrightarrow{\nu} B$ be a normalization, and $\mathbb{P}^1 \xrightarrow{\tilde{\nu}} \mathbb{P}^1$ the lifting of $\nu$ through $\nu$. The backward $\tilde{\nu}$-orbit of $\nu^{-1}(B \cap Q)$ contains at most two points, $\mathbb{P}^1 \setminus \{\text{three points}\}$ being hyperbolic. Replacing $f$ by an iterate, we may assume that the points of $\nu^{-1}(B \cap Q)$ are completely $\nu$-invariant, hence critical for $\tilde{\nu}$.

Now, for all $p \in B \cap Q$, $T_p(B) \neq T_p(Q)$. Indeed, let $(\tilde{B}, p)$ be a local irreducible component of $(B, p)$, and $(\tilde{Q}, p)$ one of $(Q, p)$. Since $g(\tilde{B}, p) = (\tilde{B}, p)$ and $d\tilde{g}(\nu^{-1}(p)) = 0$, Lemma 3.1 (2) implies that $d\tilde{g}(p)|_{T_p(B)} = 0$. Since $f(Q, p) \subset (Q, p)$ and $Q$ is elliptic, Lemma 3.3 implies that $d\tilde{g}(p)|_{T_p(Q)} \neq 0$. Consequently, $T_p(\tilde{Q}) \neq T_p(\tilde{B})$.

The local intersection number of $B$ and $Q$ at any $p \in B \cap Q$ is then $m_p(B, Q) = m_p(B)m_p(Q)$. By Bézout, $\deg(B)\deg(Q) = \sum_{p \in B \cap Q} m_p(B)m_p(Q)$. For all $p \in B$, $m_p(B) < \deg(B)$. Given two distinct points $p$ and $q$ in $B$, $m_p(B) + m_q(B) \leq \deg(B)$.
Recall that $B \cap Q$ consists of either one or two points. In both cases we immediately get a contradiction.

This means that $B$ is empty. It follows that, for some large enough $n_0$, $B \setminus Q_{n_0}$ is hyperbolic for all irreducible curves $B \subset \mathbb{P}^2$.

By Theorems 5.2 and 5.1, $\mathbb{P}^2 \setminus Q_{n_0}$ is hyperbolically embedded in $\mathbb{P}^2$. □

We will use the following simple remark, to generate computer pictures of Julia sets as basin boundaries.

**Remark 5.5.** Assume that $\mathbb{P}^2 \overset{f}{\rightarrow} \mathbb{P}^2$ leaves invariant an elliptic plane curve $Q$ and a line $L$. If $f$ has an attracting point $a \in L$ and if some point $r \in Q \cap L$ is repelling for the restriction of $f$ to $L$, then $J(f)$ equals the boundary of the basin of $a$.

**Proof.** Denote by $L \overset{\phi}{\rightarrow} L$ the restriction of $f$, and by $A$ the basin of $a$. It is clear that $J(f) \supset \partial A$. Since $r \in J(\phi)$, we deduce that $r \in \partial A$. The backward $f$-orbit of $r$ is dense in $J(f)$, hence $J(f) \subset \partial A$. □

### 6. Smooth cubics.

#### 6.1. Invariance.

Given a smooth cubic $C$, pick the group structure $\mathbb{C} \overset{[1]}{\rightarrow} (C, +, 0)$ so that $s(C_C(1)) = 0$. Three points on $C$ are collinear when their sum is 0, and 0 is a flex of $C$.

**Proposition 6.1.** Given a smooth plane cubic $C$ and a multiplier $0 \neq m \in \mathbb{Z}(C)$, $R_C(m) = \{[n] \in C : 3\frac{[m]}{[n]} = c_m\}$, hence $r_C(m) = 9|m|^2$. Given $[n] \in R_C(m)$, the self-map $C \overset{g}{\rightarrow} C$, $g[n] = [mn - n]$, admits regular extensions to $\mathbb{P}^2$. Moreover, the flexes of $C$ are pre-periodic for $g$.

**Proof.** The first statement follows from Remark 2.4 and Remark 3.4. Since $C_3 \cup g(C \setminus m \mathbb{P}^2) \subset C \setminus m \mathbb{P}^2$, the flexes of $C$ are pre-periodic for $g$. When $|m|^2 \geq 3$, the generic extension of $g$ is regular, Lemma 6.2 below concludes the proof when $|m|^2 = 2$. □

#### 6.2. Algebraic degree.

We find in this subsection the self-maps of $\mathbb{P}^2$ of algebraic degree 2 that leave invariant a smooth cubic.

**Lemma 6.2.** If a rational self-map $f$ of $\mathbb{P}^2$ with $d(f) = 2$ leaves invariant a smooth plane cubic, then $f$ is regular.

**Proof.** Let $C$ be an $f$-invariant smooth cubic, and $C \overset{g}{\rightarrow} C$ the restriction of $f$.

We see that $f$ does not contract curves. Indeed, assume that $E$ is an irreducible plane curve so that $f(E \setminus I(f))$ is a point $q \in \mathbb{P}^2$. Since $E \cap C \neq \emptyset$, $q \in C$. Since $E$ is contracted by $f$, $\deg(E) \leq d(f)$, hence $E$ is a line or a conic. Given $p \in E \cap C$, $df(p)|_{T_pE} = 0$ and $df(p)|_{T_pC} \neq 0$. Therefore, $E$ meets $C$ transversely. Since $E \cap C \subset g^{-1}(q)$, we get $2 = \deg(g) \geq \deg(E) \deg(C) \geq 3$, a contradiction.

Clearly, $f(\mathbb{P}^2 \setminus I(f))$ is Zariski dense in $\mathbb{P}^2$. (Otherwise, $f(\mathbb{P}^2 \setminus I(f)) = C$. Given a line $L$ in $\mathbb{P}^2$, $f$ would induce an isomorphism from $L$ onto $C$.)

Let $Q$ be an irreducible component of $f^*C - C$. Then $\deg(Q) \leq 3$. Since $f$ induces a surjective map from $Q$ to $C$, $Q$ is irrational, Therefore, $Q$ is a smooth cubic, and $f^*C = C + Q$. The support of $f^*C$ must contain $I(f)$, hence $Q \neq C$ and $I(f) \subset Q$. Let $C \overset{i}{\rightarrow} \mathbb{P}^2$ and $Q \overset{h}{\rightarrow} \mathbb{P}^2$ be the inclusion maps, and $Q \overset{h}{\rightarrow} C$ the restriction of $f$. Given a line $L$ in $\mathbb{P}^2$, the divisors $j^*f^*L$ and $h^*i^*L$ coincide on
$Q \setminus I(f)$. When $L \cap h(I(f)) = \emptyset$, the support of $h^*i^*L$ does not meet $I(f)$, hence $j^*f^*L = h^*i^*L + \sum_{i \in I(f)} n_i(L)(i)$, with $0 < n_i(L) \in \mathbb{Z}$.

Assume $I(f) \neq \emptyset$. Taking degrees, we get $\deg(h) = 1$ and $\sum_{i \in I(f)} n_i(L) = 3$. Now, fix a point $q \in C \setminus (f(C \cap Q) \cup h(I(f)))$, and two lines $L$ and $M$ in $\mathbb{P}^2$ that pass through $q$ and do not meet $h(I(f))$. Then $(f^*L) \cdot (f^*M) \geq g^*(q) + h^*(q) + \sum_{i \in I(f)} (i)$, hence $I(f)$ consists of one point, $I(f) = \{i\}$, and $f^*L$ and $f^*M$ meet transversely at $i$. Since $n_i(L) = 3 = n_i(M)$, we get $T_i(f^*L) = T_i(f^*M)$, hence $f^*L$ and $f^*M$ are tangent at $i$. This contradiction shows that $f$ is regular. \qed

Given $C \subset \mathbb{P}^2$, let $\text{Aut}_C(\mathbb{P}^2) := \{A \in \text{Aut}(\mathbb{P}^2) : A(C) = C\}$.

**Lemma 6.3.** Let $C$ be the set of pairs $(f, C)$, where $f$ is a regular self-map of $\mathbb{P}^2$ with $d(f) = 2$, and $C$ is an $f$-invariant smooth plane cubic. Let $\text{Aut}(\mathbb{P}^2)$ act by conjugation on $C$, $A \ast (f, C) = (AfA^{-1}, AC)$. Then card $(C/\text{Aut}(\mathbb{P}^2)) = 20$.

**Proof.** Given $\lambda$ in the Siegel figure, pick a cubic $C_\lambda \simeq \mathbb{C}(\mathbb{P}^2 \cup \mathbb{Z} \lambda)$. Fix $(f, C) \in C$. There exist $\lambda$ and $A \in \text{Aut}(\mathbb{P}^2)$ with $A(C) = C_\lambda$, hence we may assume $C = C_\lambda$. Let $F$ be the set of flexes of $C$. Pick a group structure $\mathbb{C} \xrightarrow{\mathbb{C}} (C, +, 0)$ with $[0] \in F$. Let $m := m_c(f)$ and $[n] := -f[0]$. Since $|n|^2 = 2$, $\lambda \in \{i, i\sqrt{2}, \frac{1+i\sqrt{2}}{2}\}$, and $m \in M(\lambda)$, where $M(i) = \{\pm 1 \pm i\}$, $M(i\sqrt{2}) = \{\pm i\sqrt{2}\}$, and $M\left(\frac{1+i\sqrt{2}}{2}\right) = \{\frac{1}{2} \pm i\sqrt{2}\}$. By Remark 3.4, $[n] \in N(m) := (\{\frac{1}{2}\} + F) \cup (\{\frac{1-m}{2}\} + F)$. Identify with $\mathbb{N}(m)$ the set of self-maps of $\mathbb{P}^2$ of algebraic degree $2$ that leave $C$ invariant and have multiplier $m$ on $C$. Similarly, $\text{Aut}_C(\mathbb{P}^2)$ is identified with $A : \mathbb{N}(C) \times F$, $A = (m_c(A), -A[0])$. Now, $A$ acts by conjugation on $\mathbb{N}(m)$, $(u, [v]) \ast [n] = [un + (1-m)v]$. When $\lambda = 1$, $A \ast \left(\frac{1}{2}\right) = \mathbb{N}(m)$. When $\lambda = \frac{1+i\sqrt{2}}{2}$, the orbits of $A$ on $\mathbb{N}(m)$ are $A \ast \left(\frac{1}{2}\right), A \ast \left(\frac{1}{2}\right)$.

When $\lambda = i\sqrt{2}$, the orbits of $A$ on $\mathbb{N}(m)$ are $A \ast \left(\frac{1}{2}\right), A \ast \left(\frac{1}{2}\right)$. \qed

Given $\lambda \in \{i, i\sqrt{2}, \frac{1+i\sqrt{2}}{2}\}$, $m \in M(\lambda)$, $[n] \in N(m)$ as in the proof of Lemma 6.3, let $f[m,n]$ be the extension to $\mathbb{P}^2$ of the self-map $[z] \mapsto [mn - n]$ of $C_\lambda$.

Given two points $p \neq q$ in $\mathbb{P}^2$, let $L(p, q)$ be the line passing through $p$ and $q$. As usual, $\mathbb{P}^2 \simeq \mathbb{P}^2$ denotes the space of lines in $\mathbb{P}^2$.

**Lemma 6.4.** Given a self-map $f$ of $\mathbb{P}^2$ with $d(f) = 2$ that leaves invariant a smooth cubic $C \subset \mathbb{P}^2$, let $\mathcal{L} := \{L \subset \mathbb{P}^2 : \deg(f(L)) = 1\}$, and $\mathcal{P} := \{f(L) : L \in \mathcal{L}\}$. Then $\mathcal{L}$ and $\mathcal{P}$ are smooth cubics in $\mathbb{P}^2$, and $\mathcal{L}$ is isomorphic to $C$. Moreover, $\mathcal{L}$ and $\mathcal{P}$ have the same set $\mathcal{F}$ of flexes, $\mathcal{F}$ depends only on $C$, and $\{f(L) : L \in \mathcal{F}\} = \mathcal{F}$. Finally, $\mathcal{L} = \mathcal{P}$ iff $\mathcal{P}$ is isomorphic to $C$ iff $f$ has either 2 or 4 invariant lines.

**Proof.** We may assume that $C = C_\lambda$, with $\lambda \in \{i, i\sqrt{2}, \frac{1+i\sqrt{2}}{2}\}$, and $f = f[m,n]$, with $m \in M(\lambda)$ and $[n] \in N(m)$. We see that $\mathcal{L} = \{L(p, q) : p \neq q \& f(p) = f(q)\}$, hence $\mathcal{L}$ is a curve of degree 3 in $\mathbb{P}^2$. Let $L[z] := L([z + \frac{1}{2}], [z + \frac{1+m}{2}]) = L[z + \frac{1+m}{2}]$. One of the components of $\mathcal{L}$ is the elliptic curve $\mathcal{L}_0 := \{L[z] : [z] \in C\}$, hence $\mathcal{L} \subset \mathcal{L}_0$ is a smooth cubic in $\mathbb{P}^2$. The map $\mathcal{L} \xrightarrow{\psi} C, \psi(L[z]) = [mz]$, is an isomorphism. Since $L[\frac{1}{2}] \cap L[\frac{1}{2}] = \{[0]\}, \sigma(\psi, C_\lambda(1)) = [0]$. Therefore, three lines $L[z]$, concur iff $[\sum_j z_j] \in \{[0], [m]\}$. It follows that the set of flexes of $\mathcal{L}$ is $\mathcal{F} := \{L[z] : [z] \in \mathcal{F}\}$. When $[z] \in \mathcal{F}$, $L[z] = L([z + \frac{1}{2}], [z + \frac{1}{2}])$, hence $\mathcal{F}$ does not depend on $m$ or $[n]$.
Let \( P[z] := L([z + \frac{m}{3} - n], [z + \frac{m}{3} + 2n]) \). Then \( f(L[z]) = P[mz] = f(L[z + \frac{m}{3}]) \), and \( \mathcal{P} = \{ P[z] : [z] \in C \} \), a smooth cubic in \( \mathbb{P}^2 \). Let \( C \xrightarrow{\tilde{\psi}} \tilde{C} \) be the quotient map associated to the translation \( [z] \mapsto [z + 3n] \). The map \( \mathcal{P} \xrightarrow{\tilde{\psi}} \tilde{\mathcal{C}}, \tilde{\psi}(P[z]) = \tilde{[z]} \), is an isomorphism, since \( P[n - \frac{m}{3}] \cap P[-\frac{m}{3}] \cap P(\frac{m}{3} n) = \{ 0 \} \), \( \sigma(\tilde{\psi} \circ \mathcal{P}(1)) = \{ 0 \} \). It follows that three lines \( P[z_j] \) concur if \( \sum_j z_j \in \{ 0, [3n] \} \), and then the set of flexes of \( \mathcal{P} \) is also \( F \). There are five cases to consider:

1. \( \lambda = i \) and \( [n] = [\frac{1}{2}] \),
2. \( \lambda = i \sqrt{2} \) and \( [n] \in \{ [\frac{1}{2}], [\frac{1}{2} - \frac{m}{3}] \} \),
3. \( \lambda = i \sqrt{2} \) and \( [n] \in \{ [\frac{1}{2}], [\frac{1}{2} + \frac{m}{3}] \} \),
4. \( \lambda = \frac{1 + i \sqrt{3}}{2} \) and \( [n] = [\frac{1}{2}] \),
5. \( \lambda = \frac{1 + i \sqrt{3}}{2} \) and \( [n] = [\frac{1}{2} - \frac{m}{3}] \).

For all \( z \in F \), \( P[z] = L[z + \frac{1}{3}] \) in case (3), and \( P[z] = L[z] \) in all other cases. It follows that \( f \) permutes the lines in \( F \). For further reference, let \( F \xrightarrow{f} \mathbb{P}^1 \) denote this permutation, and define \( F \xrightarrow{\phi := \psi \circ f} \), where \( [r] = [\frac{1}{3}] \) in case (3), and \( [r] = [0] \) in the other four cases. Note that \( \mathcal{L} = \mathcal{P} \) iff \( \mathcal{L} = \mathcal{P} \) iff \( \lambda = \frac{1 + i \sqrt{3}}{2} \) and \( [n] = [\frac{1}{2} - \frac{m}{3}] \), which is case (5). In this case, \( f \) has \( m + 1 \) invariant lines. In the other four cases, let \( \ell \) be the sequence of lengths of the \( f \)-cycles of lines, ordered increasingly. Then \( \ell = (1, 8) \) in the cases (1) and (4), \( \ell = (1, 1, 1, 2, 2) \) in case (2), and \( \ell = (3, 6) \) in case (3).

**Lemma 6.5.** There are no self-maps of \( \mathbb{P}^2 \) of algebraic degree 2 that leave invariant two smooth cubics.

**Proof.** We keep the notations from (the proof of) Lemma 6.4. Assume that \( f \) leaves invariant a smooth cubic \( C' \neq C \). Since \( C \simeq \mathcal{L} \simeq C' \), there exists \( A_1 \in \text{Aut}(\mathbb{P}^2) \) with \( A_1(C') = C \). Since \( A_1 f A_1^{-1} \) leaves \( C \) invariant, there is \( A_2 \in \text{Aut}_C(\mathbb{P}^2) \) with \( A_2 A_1 f A_2^{-1} A_1^{-1} = f \), where \( m' \in M(\lambda), [n'] \in N(m') \). If \( A := A_2 A_1 \), then \( A f A^{-1} = f' \) and \( A(C) \neq C \). Given \( B \in \text{Aut}_C(\mathbb{P}^2) \) with \( B f = f B \), we may replace \( A \) by \( A B \). In the cases (1), (2), (4) and (5), we have \( f M = M f \), where \( M \in \text{Aut}_C(\mathbb{P}^2), M[z] = [-z] \). In the cases (2) and (3), we have \( f T = T f \), where \( T \in \text{Aut}_C(\mathbb{P}^2), T[z] := [z - \frac{m + 1}{3}] \).

Let \( \mathcal{L}' \) be the curve of lines that are mapped to lines by \( f' \), and similarly define \( \mathcal{P}', \mathcal{F}', \psi' \) and \( \phi' \). Clearly, \( \{ A(L) : L \in \mathcal{L}' \} = \mathcal{L}' \), and \( \{ A(P) : P \in \mathcal{P} \} = \mathcal{P}' \). Define \( \mathcal{L} \xrightarrow{\alpha} \mathcal{L}', \alpha(L) \equiv A(L) \). Then \( \alpha(F) = F, \) and \( \alpha F = F' \alpha \) on \( F \). Define \( C \xrightarrow{\alpha} C \). Then \( \alpha f \alpha^{-1} = f \), and \( \beta f = \phi f' \). Write \( \beta[z] = [uz - v] \), with \( u \in U(C) \) and \( [v] \in F \). Then \( u(uz + r) - v = [m'(uz - v) + r'] \) for all \( [z] \in F \). We get \( m = m', \) hence \( \mathcal{L} = \mathcal{L}', \psi = \psi', \) and \( \{ (m - 1)v = [r' - ur] \} \). It suffices to show \( \beta = 1_\mathbb{P} \), since then \( \mathcal{L} = \mathcal{L}', \) and \( A = 1_\mathbb{P} \), contradicting \( A(C) \neq C \).

In the cases (5) and (4), \( [r'] = [0] = [r'] \), hence \( [v] = 0 \). Since \( f M = M f \), we may assume \( u = 1, \) i.e. \( \beta = 1_\mathbb{P} \).

In the cases (3) and (2), \( [r' - ur] \in \mathbb{Z}[\frac{m}{3}] \) and \( [(m - 1)v] \in \mathbb{Z}[\frac{m}{3}] \), hence \( [r' - ur] = [0] \) and \( [v] \in \mathbb{Z}[\frac{m}{3}] \). Since \( f T = T f \), we may assume \( [v] = 0 \), and then \( [r'] = [ur] \). In case (3), we get \( [r'] = [\frac{m}{3}] = [r] \) and \( u = 1 \), hence \( \beta = 1_\mathbb{P} \). In case (2), since \( f M = M f \), we may assume \( u = 1, \) i.e. \( \beta = 1_\mathbb{P} \).
In case (1), \( f = f' \), \( [r] = 0 \), \( [v] = 0 \). Since \( L[0] \) is the only \( f \)-invariant line, \( \beta[0] = [0] \). Since \( L[\frac{1}{2}] = f^*L[0] - L[0] \), \( \beta[\frac{1}{2}] = [\frac{1}{2}] \). Therefore, \( u = \pm 1 \). Since \( fM = Mf \), we may assume \( u = 1 \), i.e. \( \beta = 1_C \).

\[ \Box \]

**Proposition 6.6.** Up to conjugation by a Möbius transformation, there are 20 self-maps of \( \mathbb{P}^2 \) of algebraic degree 2 with an invariant smooth cubic.

**Proof.** Assume \( Af_m[n]A^{-1} = f_{m'}[n'] \). By Lemma 6.5, \( AC_\lambda = C_{\lambda'} \), hence \( \lambda = \lambda' \). It follows that \( m = m' \), and then \( [n] = [n'] \).

\[ \Box \]

**Remark 6.7.** Given \((f, C) \in C\), the ramification divisor \( R \) of \( f \) is a smooth cubic isomorphic to \( \mathcal{P} \), and \( R \cap C = F \). The branching curve \( B := f(R) \) is the dual of \( \mathcal{P} \).

**Proof.** Note that \( R \) is a cubic in \( \mathbb{P}^2 \). Since \( f(L[z]) = f(L[z + \frac{m}{3}]) \), the map \( C \rightarrow R, r[z] := L[z] \cap L[z + \frac{m}{3}] \), is well-defined. Its fibers are the orbits of the translation \( \tau[z] = [z + \frac{m}{3}] \), and the same is true for the map \( C \rightarrow [z] \rightarrow P[mz] \in \mathcal{P} \). Therefore, \( R \approx C/\tau \approx \mathcal{P} \). When \([z] \in F, r[z + \frac{1}{2}] = [z] \), hence \( R \cap C = F \).

Given \( x \in R \), the differential \( df(x) \) has 1-dimensional kernel \( L_x \), or else \( \mathcal{L} \) would contain the pencil of lines through \( x \). Clearly, \( L_x \in \mathcal{L} \). Let \( L \neq L_x \) be another line in \( \mathcal{L} \) that passes through \( x \). If \( x \) is not critical for the restriction of \( f \) to \( R \), then \( f(L) = T_{f[x]}B \). Therefore, \( \mathcal{P} \) is the dual of \( B \).

\[ \Box \]

**Remark 6.8.** When \( \lambda = \frac{1 + \sqrt{3}}{2} \) and \( [n] = \frac{[1 - \sqrt{3}]}{2} \), we have \( \mathcal{L} = \mathcal{P} \), i.e. \( f := f_m[n] \), \( m = \frac{1 + \sqrt{3}}{2} \), leaves invariant the smooth cubic of lines \( \mathcal{L} \), inducing on \( \mathcal{L} \) the self-map \( g(L[z]) = L[mz] \). The map \( \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{P} \), \( \pi(L_1, L_2) = L_1 \cap L_2 \), is regular, and \( \pi(g, g) = f \pi \). It follows that \( f \) is strictly critically finite, with \( f(f) = \mathbb{P}^2 \). The ramification \( R \) is isomorphic to \( C \), and the branching \( B \) is the dual of \( \mathcal{L} \). Moreover, \( f^*B = B + 2R \). Such maps with an invariant smooth cubic of lines appear in Proposition 7.2. The self-maps with an invariant curve of plane curves are discussed in [4].

**Example 6.9.** The smooth cubic \( y^2z = x(x - z)(x - b^2z) \), where \( b = 1 + \sqrt{2} \), is given in the Siegel figure by \( iv \sqrt{2} \). Pick the origin at \([0, 1, 0]\). Then \( f_{iv \sqrt{2}[1, 0, 1]} = \)
[\[y^2 + (3b + 2)x^2 - 2b^2xz - b^3z^2, -2(b + 1)yg(x + bz), y^2 - (b - 2)x^2 + 2b^2xz - b^3z^2\]. This self-map has five basins of attraction; by Remark 5.5, its Julia set equals their common boundary. Figure 1 shows the traces of the Julia set on the lines (x = 0) (left) and (x = 0.01iz) (right), near the flex [0, 1, 0].

6.3. Tangent processes. The classic “tangent process” on a smooth plane cubic C maps p \in C to the residual intersection g(p) of C with the tangent line T_p(C). Clearly, m(g) = -2, and g admits regular extensions to \mathbb{P}^2.

Definition 6.10. A tangent process on a (possibly singular) elliptic plane curve Q is a self-map of Q with multiplier -2 that admits regular extensions to \mathbb{P}^2.

In suitable coordinates, the classic tangent processes are given by Desboves’ tangent formula. Let \([x_j]\) be the coordinates in \mathbb{P}^2, j \in \mathbb{Z}/3. When \(k^3 \neq 1\), the cubic \(C_k := (h_k = 0)\), \(h_k := \sum x_j^3 - 3k \prod_j x_j\), is smooth. Every elliptic curve is isomorphic to some \(C_k\). The map \(D[x_j] := [x_j(x_j^3 + x_{j+1}^3 - x_{j+2}^3)]\) is a rational extension of the classic tangent process \(g_k\) on \(C_k\). Self-maps that leave invariant arbitrarily many of the curves \(C_k\) can be obtained by adding multiples of \(\prod_k h_k\) to the components of a large iterate of D. The extensions to \mathbb{P}^2 of \(g_k\) are parametrized by \(3 \times 3\) matrices \(A = (a_{ji})\) of complex numbers, \(D_{k, A}[x_j] = [x_j(x_j^3 + x_{j+1}^3 - x_{j+2}^3) + (\sum_i a_{ji} x_j) h_k]\). By Remark 2.5, \(I(D)\) can be used to get regular extensions of \(g_k\) with attracting points.

Example 6.11. When the matrix \(A := \text{diag}(a_j)\) is diagonal, the map \(D_{k, A}\) commutes with \([x_j] \mapsto [\exp\left(\frac{2\pi i}{3}\right) x_j]\), and the lines \((x_j = 0)\) are \(D_{k, A}\)-invariant. If \(\max(|a_{j+1} - 1|, |a_{j+2}| + 1) < |a_j|\), the fixed point \(P_j := (x_{j+1} = 0 = x_{j+2})\) is attracting. If, further assumed, \(\max(|2 + 3(a_j - a_{j+1})|, |2 + 3(a_{j+2} - a_j)|) > 1\), Remark 5.5 implies that \(J(D_{k, A})\) is the boundary of the basin of \(P_j\). For example, when \(A := \text{diag}(i, 2i - 1, 1 - 2i)\), the Julia set \(J(D_{k, A})\) is the common boundary of the basins of \([0, 1, 0]\) and \([0, 0, 1]\). Figure 2 shows the trace of \(J(D_{0, A})\) on the line \((x + y = 0)\) near \([1, -1, 0]\) (left), and a zoom-in at the center (right).

When \(A = \text{diag}(a, 1, -1), a \neq 0\), \(D_{0, A}\) leaves invariant the Fermat cubic \(C_0\) and the pencil of lines through \([1, 0, 0]\). We discuss such maps in the next subsection.

FIGURE 2. Tangent process on the Fermat cubic.
6.4. Elementary maps with an invariant smooth cubic. Given $P \in \mathbb{P}^2$, a map $\mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2$ with $d(f) > 1$ is elementary with center $P$ iff it leaves invariant the pencil $\mathcal{P}$ of lines passing through $P$, i.e. $f(L) \in \mathcal{P}$ for all $L \in \mathcal{P}$.

Lemma 6.12. Let $C$ be a smooth elliptic curve, and $C \xrightarrow{q} \mathbb{P}^1$ a non-constant elliptic function. Assume that a self-map $\mathbb{P}^1 \xrightarrow{g} \mathbb{P}^1$ with $\deg(g) > 1$ lifts through $q$ to a self-map $C \xrightarrow{h} C$, $qh = qg$. Then there exists a smooth elliptic curve $D$, a map $C \xrightarrow{\alpha} D$, and a finite nontrivial group $G < \text{Aut}(D)$, with associated quotient map $D \xrightarrow{\beta} D/G = \mathbb{P}^1$, so that $q = \alpha \beta$.

Proof. Let $B \subset \mathbb{P}^1$ be the branch locus of $q$, and $R = q^{-1}(B)$. Then $C \setminus R \xrightarrow{q} \mathbb{P}^1 \setminus B$ is an unramified covering, of degree $n + 1$. Let $m := m(h)$. Since $|m| > 1$, we can find an $h$-periodic point $c_0 \in C \setminus R$. Let $c_i$, $1 \leq i \leq n$, be the other points in the fiber $q^{-1}(q(c_0))$. Replacing $g$ by an iterate, we may assume that $h(c_0) = c_0$ and $h^2(c_i) = h(c_i)$. Let $(C, c_0) \xrightarrow{\gamma_i} (C, c_i)$ be the transition maps on the sheets of $q_i$, i.e. $q\gamma_i = q$. Since $h^2\gamma_i(c_0) = h\gamma_i h(c_0)$ and $q(h^2\gamma_i) = q(h\gamma_i h)$, we get $h^2\gamma_i = h\gamma_i h$.

Fix a universal covering $\mathcal{C} \xrightarrow{u} C$ with $u(0) = c_0$. Then $h$ lifts through $u$ to the linear map $\mathcal{C} \xrightarrow{h} \mathcal{C}$, $H(z) = m z$. Choose $z_i \in u^{-1}(c_i)$, and let $(\mathcal{C}, 0) \xrightarrow{\Gamma_i} (\mathcal{C}, z_i)$ be the lifting of $\gamma_i$ through $u$. Let $\Gamma_i(z) = z_i + m_i z + \sum_{j > 2} m_{ij} z^j$ be the Taylor series at 0. Since $|m| > 1$ and $uH^2\Gamma_i = uH\Gamma_i H$ near 0, we get $m_{ij} = 0$ for all $j \geq 2$.

Therefore, $\mathcal{C} \xrightarrow{\Gamma_i} \mathcal{C}$ are affine, $\Gamma_i(z) = m_i z + z_i$. Let $\mathcal{C} \xrightarrow{1} \mathcal{C}$ be the identity map.

Identify the lattice $\Omega = u^{-1}(c_0)$, and the group of translations on $\mathcal{C}$, $C$ with $\mathbb{C}/\Omega$, and $\mathcal{C} \xrightarrow{u} C$ with the quotient map associated to the action of $\Omega$ on $\mathcal{C}$.

Consider the map $\mathcal{C} \xrightarrow{Q} \mathbb{P}^1$, $Q = qu$, and the set $A = \{C \xrightarrow{\Gamma} \mathcal{C} : Q \Gamma = Q\}$. Clearly, $A = \bigcup_{0 \leq i \leq n} \Omega \Gamma_i$ and this union is disjoint. Therefore, $A$ is a group of affine self-maps of $\mathcal{C}$, and $\Omega$ is a subgroup of $A$ of index $[A : \Omega] = n + 1$. Note that $A(0) = Q^{-1}(Q(0))$, and that the evaluation map $A \ni \Gamma \mapsto \Gamma(0) \in A(0)$ is bijective.

The translations in $A$ form a normal subgroup $T$ of $A$, with $\Omega \leq T$. Since $Q$ is non-constant, $T \subset A(0)$ is discrete in $\mathcal{C}$, hence $T$ is a lattice in $\mathcal{C}$. Consider the elliptic curve $D = \mathcal{C}/T$, with quotient map $\mathcal{C} \xrightarrow{v} D$. Let $C \xrightarrow{\alpha} D$ be the map induced by $v$ through $u$, $\alpha = vu$. The group $G = A/T$ is a finite group of automorphisms of $D$, of order $k \leq n + 1$. Let $D \xrightarrow{\beta} D/G$ be the quotient map.

Clearly, $Q$ induces through $v$ a map $D \xrightarrow{Q} \mathbb{P}^1$, $Q = \alpha v$. We see that $\deg(\alpha) = k$. (Indeed, $\alpha^{-1}(Q(0))$ has cardinality $k$, and $Q(0)$ is not a branch point of $\alpha$.) Clearly, $\alpha$ induces through $s$ a map $D/G \xrightarrow{\beta} \mathbb{P}^1$, $\beta s = \alpha$. Since $\deg(\alpha) = k = \deg(s)$, $\beta$ is an isomorphism. We identify, via $\beta$, the map $\alpha$ with the quotient map $s$. We get $qu = Q = \alpha v = su = sru$, hence $q = sr$.

If, in Lemma 6.12, $\deg(g)$ is a prime number, then $r$ is an isomorphism; in this case, we may assume that $D = C$ and $q = s$.

The Fermat cubic $C_0 := \{x^3 + y^3 + z^3 = 0\}$ is the only smooth elliptic curve with an automorphism $\gamma$ satisfying $\gamma^3 = 1$ and $\text{Fix}(\gamma) \neq \emptyset$. Let $c_0 = [0, -1, 1]$. Then $\text{Aut}_c(C_0) \simeq \mathbb{Z}/6$, generated by $\gamma_0[z, y, z] := [\tau z, z, y]$, with $\tau = \exp(\frac{2\pi i}{3})$.

Let $G_0 = < \gamma_0 >$. Note that $\text{Fix}(G_0) = C_0 \cdot (x = 0)$. For $c \in C \setminus (x = 0)$, $G_0(c)$ consists of three collinear points that determine a line passing through $P_0 := [1, 0, 0]$. In other words, the quotient map $C_0 \xrightarrow{\alpha} C_0/G_0$ can be identified with the map
$C_0 \xrightarrow{\gamma} \hat{P}_0$ that associates to $c \in C_0$ the line joining $P_0$ to $c$. In homogeneous coordinates, $C_0 \xrightarrow{\gamma} \mathbb{P}^1$ is the central projection $\gamma_0 [x, y, z] = [y, z]$.

Put $\Omega_0 = \mathbb{Z} + \mathbb{Z} r$, and let $C \xrightarrow{\gamma} \mathbb{C}/\Omega_0$ denote the quotient map. Fix an isomorphism $\mathbb{C} \simeq \mathbb{C}/\Omega_0$ with $[0] = \gamma_0$. Then $\gamma_0 [t] = [-r t]$, as follows from differentiating $\gamma_0$ at its fixed point $\gamma_0$. Note that $\text{Fix}(G_0) = \{ z \mathbb{Z} \}$.

Proposition 2.3 yields the following criterion for a self-map of $C_0$ to admit an elementary extension with center $P_0$.

**Corollary 6.13.** Given a map $C_0 \xrightarrow{h} C_0$ with $d := \deg(h) > 1$, $h$ extends to an elementary map with center $P_0$ $\iff h(c_0) \in \text{Fix}(G_0)$ $\iff h$ commutes with $G_0$.

**Proof.** Since $m(h) \in \Omega_0 = \mathbb{Z} + \mathbb{Z} r$, either $\frac{d}{3} \in \mathbb{Z}$ or $\frac{d+1}{3} \in \mathbb{Z}$, hence $d \geq 3$.

If $h$ extends to an elementary map with center $P_0$, it induces through $\gamma_0$ a map $\mathbb{P}^1 \xrightarrow{\varphi} \mathbb{P}^1$, $g h = g' \gamma_0$. The critical points of $\varphi$ are the three fixed points of $G_0$. Since $h$ is unramified and $c_0$ is critical for $\gamma_0$, $h(c_0)$ is critical for $\varphi$, i.e. $h(c_0) \in \text{Fix}(G_0)$.

Assume $h(c_0) \in \text{Fix}(G_0)$. For all $\gamma \in G_0$, $h c_0 \gamma(c_0) = \gamma h(c_0)$. Since $m(h \gamma) = m(\gamma h)$, $h \gamma = h \gamma$. Therefore, $h$ commutes with $G_0$.

If $h$ commutes with $G_0$, it induces through $\varphi$ a self-map $\mathbb{P}^1 \xrightarrow{g} \mathbb{P}^1, g = [g_1, g_2]$, $g_j \in \mathbb{C}[y, z]$. Then $h^* \mathcal{O}_C(1) \simeq h^* q^* \mathcal{O}_{\mathbb{P}^2}(1) \simeq q^* g \mathcal{O}_{\mathbb{P}^2}(1) \simeq q^* \mathcal{O}_{\mathbb{P}^2}(d) \simeq \mathcal{O}_C(d)$.

By Proposition 2.3, $h$ extends to $\mathbb{P}^2 \xrightarrow{\varphi} \mathbb{P}^2$ with $g = [g_0, g_1, g_2, f_j \in \mathbb{C}[x, y, z]$.

We have $q^* g(y = 0) = q^* (g_1) = 0 = C_0 \cdot (g_1 = 0)$, and $h^* q^*(y = 0) = h^* (C_0 \cdot (g_1 = 0)) = C_0 \cdot (f_1 = 0)$, hence the divisors $(g_1 = 0)$ and $(f_1 = 0)$ leave the same trace on $C_0$. Therefore, there exists a constant $0 \neq \alpha_1 \in \mathbb{C}$ so that $f_1 - \alpha_1 g_1$ vanishes on $C_0$. Let $e = x^2 + y^3 + z^3$. Then there exists a polynomial $\beta_1 \in \mathbb{C}[x, y, z]$ so that $f_1 - \alpha_1 g_1 = e \beta_1$. Similarly, there exist a constant $0 \neq \alpha_2 \in \mathbb{C}$ and a polynomial $\beta_2 \in \mathbb{C}[x, y, z]$ so that $f_2 - \alpha_2 g_2 = e \beta_2$.

Consider the rational self-map $\tilde{f}$ of $\mathbb{P}^2$, $f = [g_0, \alpha_1 g_1, \alpha_2 g_2]$. Then $\tilde{f} \subseteq \{ c_0 \}$, and $\tilde{f}|_{C_0} = f|_{C_0} = h$. When $f_0(c_0) \neq 0$, $\tilde{f}$ is an elementary extension of $h$. When $f_0(c_0) = 0$, $[f_0 + x^2 + 4 e \alpha_1 g_1, \alpha_1 g_2]$ is an elementary extension of $h$.

**Proposition 6.14.** Let $\mathbb{P}^2 \xrightarrow{\varphi} \mathbb{P}^2$ be an elementary map with center $P$, that leaves invariant a smooth cubic $C$. Then there exists a Möbius map $M \in \text{Aut}(\mathbb{P}^2)$ with $M(P) = P_0$ and $M(C) = C_0$.

**Proof.** Let $C \xrightarrow{h} C$ be the restriction of $f$ to $C$. Clearly, $P \notin C$. Let $C \xrightarrow{\omega} \hat{P}$ be the regular map that associates to a point $c \in C$ the line joining $P$ to $c$. Then $\omega$ has topological degree 3, and $h$ induces through $\omega$ a map $\mathbb{P}^1 \xrightarrow{\varphi} \mathbb{P}^1, q h = g q$. By Lemma 6.12, there exists a group $\mathbb{Z}/3 \simeq G < \text{Aut}(C)$, with $\text{Fix}(G) \neq \emptyset$, whose orbits lie on lines through $P$. Fix $c \in \text{Fix}(G)$, necessarily a flex of $C$. Pick an isomorphism $C \xrightarrow{m} C_0$ with $m(c) = c_0$. As in the proof of Proposition 2.3, $m$ extends to a Möbius map $M \in \text{Aut}(\mathbb{P}^2)$. Since $\mathbb{Z}/3 = \text{M}G^{-1} < \text{Aut}_{\gamma_0}(C_0)$, $\text{M}G^{-1} = G_0$. Since $M$ maps lines to lines and $G$-orbits to $G_0$-orbits, $M(\hat{P}) = \hat{P}_0$.

**Example 6.15.** If a map $C_0 \xrightarrow{h} C_0$ with $\deg(h) = 3$ (minimal) has an elementary extension $\mathbb{P}^2 \xrightarrow{\varphi} \mathbb{P}^2$ with center $P_0$, then $m(h) \in \{ -1, \tau - \tau_1, \tau^2 - 1 \}$. We see that $h$ maps the set of flexes of $C_0$ onto $\text{Fix}(G_0)$, and is constant on $\text{Fix}(G_0)$. Therefore, up to Möbius conjugation, there are six one-parameter families of elementary self-maps of $\mathbb{P}^2$ of algebraic degree 3, that leave invariant a smooth cubic.
By Desboves’ secant formula, \( h[t] = [(\tau - 1)t] \) is given in projective coordinates by \( h[x, y, z] = [(\tau - 1)xy, z^3 - \tau y^3, y^3 - \tau z^3] \). The elementary extensions \( f_a \) of \( h \) are obtained by adding \( a(x^3 + y^3 + z^3) \), with \( a \neq 0 \), to the first component of \( h \).

Let \( S(f) \) be the support of the Green measure associated to a self-map \( \mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2 \), and \( R(f) \) the closure of the set of repelling periodic points of \( f \).

Assume \( \mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2 \) is elementary with center \( P \). Since \( f^{-1}(P) = P \), \( P \) is super-attracting for \( f \), and the basin of attraction of \( P \), denoted \( \mathcal{A}(f) \), is connected.

**Proposition 6.16.** If an elementary map \( \mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2 \) with center \( P \) leaves invariant a smooth cubic, then:

1. \( J(f) = S(f) = R(f) = \partial \mathcal{A}(f) \), and \( \overline{\mathcal{A}(f)} = \mathbb{P}^2 \).
2. \( \bigcup_{r \geq 0} f^r(U) = \mathbb{P}^2 \setminus \{P\} \), for every open set \( U \) with \( U \cap R(f) \neq \emptyset \) and \( P \notin U \).

**Proof.** Let \( C \longrightarrow h \longrightarrow C \) be the restriction of \( f \) to an \( f \)-invariant smooth cubic \( C \), \( P \) the center of \( f \), \( C \longrightarrow q \longrightarrow \mathbb{P}^1 \) the central projection from \( P \), and \( \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \) the self-map induced by \( h \) through \( q \). Let \( m = m(h) \) and \( d = |m|^2 > 1 \). By Lemma 6.12, \( q \) is the quotient map \( C \longrightarrow C/G \) associated to a group \( \mathbb{Z}/3 \cong G \subset \text{Aut}(C) \).

Since \( g \) is strictly critically finite, \( J(g) = \mathbb{P}^1 \), and \( \mathcal{A}(f) \) is the only Fatou component of \( f \), hence \( \overline{\mathcal{A}(f)} = \mathbb{P}^2 \) if \( J(f) = \partial \mathcal{A}(f) \).

For every elementary self-map of \( \mathbb{P}^2 \), \( R(f) = S(f) \) (cf. [3]). By Corollary 6.13, \( h \) commutes with \( G \). Replacing \( f \) by an iterate, we may assume that \( f \) fixes a flex \( x \in \text{Fix}(G) \). The line \( F_x \) joining \( x \) to \( P \) is tangent to \( C \), hence \( f_x'(x) = m \), where \( F_x \xrightarrow{f} F_x \) is the polynomial function induced by \( f \). Also, since \( g^*q(x) = 3x \), we get \( g'(q(x)) = m^3 \). It follows that \( x \) is repelling for \( f \) (with eigenvalues \( m \) and \( m^3 \)), hence \( x \in S(f) \). Since \( S(f) \) is completely \( f \)-invariant, Theorem 5.4 implies \( J(f) \subset S(f) \), hence \( J(f) = S(f) \).

Clearly, \( \partial \mathcal{A}(f) \subset J(f) \). Since \( x \) is repelling for \( f_x \), we have \( x \in \partial \mathcal{A}(f_x) \subset \partial \mathcal{A}(f) \). We get \( J(f) \subset \partial \mathcal{A}(f) \), hence \( J(f) = \partial \mathcal{A}(f) \), and the first statement is proved.

Let \( F_1 \) be the non-minimal Hirzebruch surface. Statement (2) follows from [3], once we show that \( \mathbb{P}^2 \xrightarrow{f} F_1 \) has no completely invariant curves besides the negative section. Otherwise, \( f \) leaves completely invariant some affine section, so that \( f_x \) has two exceptional points. But then \( |f_x'(x)| = d \), in contradiction with \( f_x'(x) = m \).

7. DUAL OF SMOOTH CUBIC.

Given \( p \in \mathbb{P}^2 \), \( \hat{p} := \{L \in \mathbb{P}^2 : p \in L\} \) is a line in \( \hat{\mathbb{P}}^2 \). Given a line \( l \) in \( \mathbb{P}^2 \), \( \hat{l} := \bigcap_{L \ni l} L \) is a point in \( \hat{\mathbb{P}}^2 \). For every \( p \in \mathbb{P}^2 \), \( \hat{p} = p \). For every line \( l \) in \( \mathbb{P}^2 \), \( \hat{l} = l \).

Given a smooth cubic \( C \) in \( \mathbb{P}^2 \), with set \( \mathcal{F} \) of flexes, choose the group structure \( \mathbb{C} \xrightarrow{1} \mathbb{C}/\mathbb{Q} = C \) so that \( \{0\} \in \mathcal{F} \). Given \( m \in \mathbb{Z} \), let \( C_m := \{c \in C : mc = 0\} \).

The dual curve \( \hat{C} \), defined as the curve of tangents to \( C \), is a sextic in \( \hat{\mathbb{P}}^2 \) with ordinary cusps \( T_a \hat{C} \), \( a \in \mathcal{F} \), normalized by \( C \xrightarrow{\nu} \hat{C} \), \( \nu(c) := T_c(C) \). For all \( c \in C \), \( T_{\nu(c)} \hat{C} = \hat{c} \). In this way, the dual of \( \hat{C} \), defined as the curve of tangents to \( \hat{C} \), is identified with \( C \). Given a line \( l \) in \( \mathbb{P}^2 \) not passing through the cusps of \( \hat{C} \), recall that \( l_C := l^*l \), where \( \hat{C} \xrightarrow{\nu^*} \hat{\mathbb{P}}^2 \) denotes the inclusion map, and \( l_C := \nu^*l \hat{C} \). We see that \( l_C = \sum \{(c) : l \in T_c(C)\} \). For generic \( [c] \in C \), \( [c]_C = 2([c]) + \sum [\hat{b} - \frac{\hat{b}}{2}] \).
Proposition 7.1. For all $0 \neq m \in \mathbb{Z}(C)$, $R_C(m) = \mathcal{F}$.

Proof. This follows from Proposition 7.2 below. When $m \in \mathbb{Z}$, we give a proof based on Theorem 3.14. The condition that $[ma - n] \in \mathcal{F}$ for all $[a] \in \mathcal{F}$ means that $[n] \in \mathcal{F}$. Fix $[a] \in \mathcal{F}$. For generic $[c] \in C$, we calculate $a(a) = 2\zeta(a-c) + \sum_{b \in \mathbb{Z}} \zeta(a+b)$. Since $\sum_{b \in \mathbb{Z}} \zeta(z-b) = 2\zeta(2z)$ for all $[z] \in C$, we get $a(a) = 2\zeta(a-c) + 2\zeta(2a+c) = 2\eta(3a)$. If $[n] \in \mathcal{F}$, then $\alpha_{m,[a]}[a] = 2m\eta(3ma - 3n) - 2m^2\eta(3a) + \eta(6mn) = 0$. □

The condition $[n] \in \mathcal{F}$ means that the map $C \stackrel{h}{\to} C$, $h[z] = [mz - n]$, preserves the collinearity on the smooth cubic $C$: if $[a]$, $[b]$, $[c]$ are three collinear points on $C$, then $h[a]$, $h[b]$, $h[c]$ are also collinear. The following construction is apparently due to T. Ueda; the second author has learned it from M. Jonsson.

Proposition 7.2. Let $C \times C \to \mathbb{P}^1$, $\pi(c_1, c_2) := L(c_1, c_2)$. Note that $\pi(c, c) = \nu(c)$ for all $c \in C$. If $C \stackrel{h}{\to} C$ preserves the collinearity on $C$, then $(h, h)$ induces through $\pi$ a regular self-map $\tilde{h}$ of $\mathbb{P}^1$, $\tilde{h} = \pi(h, h)$. Moreover,

1. The branching curve of $\tilde{h}$ is $\tilde{C}$, and $\tilde{h}(\tilde{C}) = \mathbb{P}^1$.
2. The ruled surface $\{ (c, L) \in C \times \mathbb{P}^2 : c \in L \}$ is $(h, h)$-invariant.
3. The dual $\tilde{C}$ of $C$ is $h$-invariant, with $h(h) = h$.
4. The dual $\tilde{C}$ of $C$ is $h$-invariant; for all $c \in C$, $\tilde{h}(c) = (h(c))^\ast$.

Proof. The map $\pi$ can be identified with the quotient map associated to the action on $C \times C$ of the symmetric group $\Sigma_3$ generated by the involution $\alpha(a, b) = (b, a)$ and the 3-cycle $\beta(a, b) = (b, a - b)$. Since $(h, h)$ commutes with $\alpha$ and $\beta$, it induces a regular self-map $\tilde{h}$ of $\mathbb{P}^2$. The properties of $\tilde{h}$ are easy to verify. □

Corollary 7.3. The curve $\tilde{C}$ admits nine tangent processes. They extend uniquely to $\mathbb{P}^2$. One of them fixes some (all) of the nine cusps of $\tilde{C}$.

Example 7.4. Let $C_k = (\sum_j x_j^k)$, as in subsection 6.3. The normalization of $\tilde{C}_k$ is $\nu_k[x_j] = [x_j^2 - kx_{j+1}x_{j+2}]$. When $k = 0$, $\tilde{C}_0 = \left(2 \sum a_j^2 = (\sum_j a_j^2)^2 \right)$. The Desboves map $g_0$ on $C_0$ induces through $\nu_0$ the tangent process on $\tilde{C}_0$ that fixes the cusps. We calculate $g_0[a_j] = [a_j(-3a_j^2 + 2 \sum a_j^2)]$.

8. Elliptic quartics with two singular points.

8.1. Normalization. Given two plane curves $Q_1$ and $Q_2$, we write $Q_1 \sim Q_2$ when there exists a Möbius transformation $M \in \text{Aut}(\mathbb{P}^2)$ with $M(Q_1) = Q_2$. The elliptic quartics with two singular points can be represented as follows.

Proposition 8.1. Let $\mathbb{C} \to \mathbb{C}/\mathbb{Z} = C$ be an elliptic curve. Given $(a, b, \bar{a}, \bar{b}) \in \mathbb{C}^4$, let $\tilde{b} := -a - b - \bar{a}, \bar{b} := -2b - \bar{b}$, and $c := -2n + b$. Assume that $[a] \neq [\bar{a}], [a] \neq [\bar{b}], [b] \neq [\bar{a}], [b] \neq [\bar{b}], [\bar{a}] \neq [\bar{b}], [\bar{a}] \neq [b], [\bar{a}] \neq [\bar{b}]$, and $[a + b] \neq [\bar{a} + \bar{b}]$. Define $\Psi : \mathbb{C}^5$, $\Psi = (X, Y, Z)$, by: $X(t) = \sigma^2(t - \bar{a})\sigma(t - \bar{b})\sigma(t - \bar{c})$, $Y(t) = \sigma^2(t - a)\sigma(t - b)\sigma(t - c)$, $Z(t) = \sigma(t - a)\sigma(t - b)\sigma(t - \bar{a})\sigma(t - \bar{b})$. 

Theorem 8.2. For all $0 \neq m \in \mathbb{Z}(C)$, $R_C(m) = \mathcal{F}$.
Then $\Psi$ induces through $C \xrightarrow{\nu} \mathbb{P}^2$ a normalization $C \xrightarrow{\nu} Q$, $\nu[t] = \mathbb{P}[\Psi(t)]$, of an elliptic quartic $Q := C(a, b, \delta)$ with two ordinary singularities, $q := [1, 0, \delta], \tilde{q} := [0, 1, 0]$, and $s(\nu^*\mathcal{O}_Q(1)) = 0$, $\nu^*q = ([a]) + ([b]), \nu^*\tilde{q} = ([\tilde{a}]) + ([\tilde{b}])$.

All elliptic quartics with two singularities are Möbius images of such $C(a, b, \delta)$.

**Proof.** It is clear that $X(t), Y(t)$ and $Z(t)$ do not vanish simultaneously. For $\omega \in \Omega$, $X(t) = Y(t) = Z(t) = \exp(2(t + \omega)\eta(\omega))$, so $C \xrightarrow{\nu} \mathbb{P}^2$ is well-defined. In affine coordinates, $x[t] = \frac{X(t)}{Z(t)}$ and $y[t] = \frac{Y(t)}{Z(t)}$. We show that $\nu$ is an injective on $C \setminus \nu^{-1}(Z = 0)$. Indeed, if $\nu[t_1] = \nu[t_2] \neq (Z = 0)$ for some $t_1 \neq t_2$, then $[t_1] + [t_2]$ would be a fiber of both $x[t_1]$ and $y[t_1]$, hence $[a] + [b] = [t_1] + [t_2] = [\tilde{a}] + [\tilde{b}]$. Therefore, $C \xrightarrow{\nu} Q \subset \mathbb{P}^2$ is a normalization map. Clearly, $\nu^*(q) = ([a]) + ([b]), \nu^*(\tilde{q}) = ([\tilde{a}]) + ([\tilde{b}])$, and $s(\nu^*\mathcal{O}_Q(1)) = s((Z = 0)_C) = 0$. In local coordinates, we see that $q$ and $\tilde{q}$ are ordinary singularities.

Given any elliptic quartic $Q$ with singularities $q \neq \tilde{q}$, fix a normalization $C \xrightarrow{\nu} Q$. Note that $m_Q(q) = 2 = m_Q(\tilde{q})$, and $L(q, \tilde{q})_C = \nu^*(q) + \nu^*(\tilde{q})$. Choose the group structure $C \xrightarrow{\nu} \mathbb{C}$ so that $\nu^*(q) = [a] + [b]$, $\nu^*(\tilde{q}) = ([\tilde{a}]) + ([\tilde{b}])$, with $a + b + \tilde{a} + \tilde{b} = 0$. Choose coordinates in $\mathbb{P}^2$ so that $L(q, \tilde{q}) = (Z = 0), T_{\tilde{q}}[a] = (X = 0)$, and $T_{\tilde{a}}[a] = (Y = 0)$. Then $(Z = 0)_C = ([a]) + ([b]) + ([\tilde{a}]) + ([\tilde{b}]), (X = 0)_C = 2([a]) + ([\tilde{b}]) + ([2a - \tilde{b}]), (Y = 0)_C = 2([a]) + ([b]) + ([2a - \tilde{b}])$. Rescaling $x$ and $y$, we get $Q \sim C(a, b, \delta)$. \[\square\]

**Remark 8.2.** Let $Q = C(a, b, \delta)$ be an elliptic quartic, with $\tilde{b} := -a - b - \delta$. Then $Q \sim C(\tilde{a}, \tilde{b}, a) \sim C(b, a, \delta) \sim C(a, b, \tilde{b}), Q \sim C(a + \omega, b + \lambda, \delta + \omega)$ for all $(\omega, \lambda, \omega) \in \Omega^3$, and $Q \sim C(a + n, b + n, m\delta - n)$ for all $(m, n) \in U(C) \times C_4$.

**Proof.** The first equivalences follow immediately from Proposition 8.1. For the one, since $\frac{m}{\sigma(t)} = m$ for all $t \in \mathbb{C}$, there exist constants $k_i \neq 0$ so that the automorphism $(x, y) \mapsto (k_1 x, k_2 y)$ maps $Q$ onto $C(a + n, b + n, m\delta - n)$. \[\square\]

**Remark 8.3.** When $q$ is a node of $Q := C(a, b, \delta)$, we have $\alpha(a, b) = \frac{\sigma(a - b)}{\sigma(a - b)}$. When $q$ is a cusp of $Q$, $\alpha(a) = \zeta(a - \tilde{a}) + \zeta(a - \tilde{b})$. Similarly for $\tilde{q}$. \[\square\]

When $\text{Aut}(Q)$ is large, the invariants $\alpha_{m[n]}$ can be calculated more explicitly.

8.2. **Invariant nodal quartics.** A node of a plane curve is *inflectional* iff it is a flex on each of its branches. A Cassini curve is an elliptic quartic with two inflectional nodes. In this subsection, we consider a 2-dimensional space of elliptic quartics with two nodes, that contains the Cassini curves.

**Remark 8.4.** Given an elliptic quartic $Q$ with two nodes, $\text{Aut}(Q) = \text{Aut}_Q(\mathbb{P}^2)$. Moreover, $\rho_Q(1) > 1$ iff $Q \sim C(a, a - \frac{1}{2}, -a)$, with $[\lambda] = 0 \neq [\delta]$ and $4[\lambda] \neq 0$. Finally, $Q$ is a Cassini curve iff $Q \sim C(\frac{\lambda}{8}, -\frac{3\lambda}{8}, -\frac{\lambda}{8})$, with $[\lambda] = 0 \neq [\delta]$. 

**Proof.** We may assume that $Q \sim C(a, b, \delta)$. As usual, $\tilde{b} := -a - b - a$. Let $g \in \text{Aut}(Q)$ be induced through $\nu$ by $C \xrightarrow{h} H, h[t] = [t - n]$. Since $h$ leaves invariant the set $\{[a], [b], [\tilde{a}], [\tilde{b}]\}$, we get $4[n] = 0$. By Remark 8.2, $g \in \text{Aut}_Q(\mathbb{P}^2)$.

When $m = 1$ and $n \neq 0$, replacing if necessary $g$ by $g^2$ we may assume that $[a - n] = [b], [b - n] = [a], [\tilde{a} - n] = [\tilde{b}]$. We get $2[n] = 0, b = a - \frac{3}{8}$ with $[\lambda] = 0$, and $\tilde{a} = a - \frac{3}{8}$ with $[\omega] = 0$. Let $a' := a - \frac{3}{8}$. By Remark 8.2, $Q \sim C(a', a' - \frac{1}{8}, -a')$. 

When $Q$ is Cassini, $0 = [3a + b] = [3b + a] = [3\bar{a} + \bar{b}] = [3\bar{b} + \bar{a}]$. We get $8[\alpha] = 0$, $[b] = -3[a]$, $2[a + \bar{a}] = 0$. By Remark 8.2, we may assume that $a = \frac{1}{2}$ with $|\lambda| = 0$, $b = -3a$, and $\bar{a} = -a + \frac{3}{2}$ with $|\omega| = 0$. As before, we may assume that $\omega = 0$. 

**Proposition 8.5.** Let $Q = C(a, a - \frac{1}{2}, -a)$, as in Remark 8.4. If $0 \neq m \in 2\mathbb{Z}$, then $R_Q(m) = \{[n] \in C_{4m} : \det(4mn, \lambda) \in 2\mathbb{Z}\}$, hence $r_Q(m) = 8m^2$.

**Proof.** By Remark 8.3, $\alpha(a, a - \frac{1}{2}) = -\exp \left(\frac{(2a \pm \frac{1}{2})\eta(\lambda)}{2}\right)$. Fix $[n] \in C_{4m}$. By Remark 3.4, $\alpha(ma - n, ma - n - \frac{\eta(\lambda)}{m}) = \exp \left(\frac{(2m^2a - 2mn - \frac{1}{2}m^2\lambda)\eta(\lambda)}{2}\right)$. Define $\delta := \det(4mn, \lambda)$. It follows that $\alpha_{m,n}(a, a - \frac{1}{2}) = \exp(\pi i \delta)$. Similarly, we calculate $\alpha_{m,n}(-a, -a + \frac{1}{2}) = \exp(-\pi i \delta)$. By Theorem 3.14, $[n] \in R_Q(m)$ iff $\delta = 2\mathbb{Z}$. 

**Example 8.6.** Given a Cassini quartic $Q$, there exists a complex number $0 \neq k \neq 1$ so that $Q \sim Q_k := (q_k = 0)$, where $q_k := (1 - x^2)(1 - y^2) - (1 - k)$. Consider the self-map $g$ of $Q_k$ that associates to $p \in Q_k$ the residual intersection $g(p)$ of $Q_k$ with the conic $\Gamma$ that passes through $q, \bar{q}$ and $p$, with tangents $T_q(\Gamma) = \{y = 1\}$ and $T_p(\Gamma) = T_{\bar{q}}(Q_k)$. Proposition 8.5 ensures that $g$ is a tangent process on $Q_k$. We calculate $g(x, y, z) = [2xy(z^2 + y^2 - 2kz^2), x^4 - y^4, -2xy(x^2 - y^2)]$. To obtain regular extensions $f$ of $g$ to $\mathbb{P}^2$, we can add $aq_k$ to the last component of $g$, with $0 \neq a \neq \pm 4i$. In the “limit” situation $k = 0 = a$, we get the degenerate map $(x, y) \mapsto \left(-\frac{x^2 + y^2}{x^2 - y^2}, -\frac{x^2 + y^2}{2xy}\right)$, whose image is the rational quartic $(x^2 + y^2 = x^2 y^2)$. When $a^2 = 8k$, the restriction of $f$ to the invariant line $(x = 0)$ is critically finite. When $a$ and $k$ are real, $\mathbb{P}^2$ is $f$-invariant, Figure 3 shows the traces on $\mathbb{R}^2$ (with center at $(0, 0)$) of the basins of attraction of $(0, 0)$ (black) and $(0, \frac{4}{2})$ (grey), for $a = \sqrt{8k}$, with $k = 0.125$ (left) and $k = 0.001$ (right). Computer experiments suggest that $J(f)$ equals the boundary of the basin of $(0, 0)$. However, the unbounded component of $Q_k \cap \mathbb{R}^2$ seems to have a basin of attraction in $\mathbb{R}^2$ (white in Figure 3).

**8.3. Quartics with a cusp and a node.** Let $C^* := C/\Omega_r$, with $\Omega_r := \mathbb{Z} \oplus \mathbb{Z}r$. Given a basis $(\lambda, \omega)$ in $\Omega_r$, let $C^*_{\lambda, \omega} := C^*/(\lambda \pm \omega, \lambda \pm \omega, \lambda \pm \omega)$. 

**Remark 8.7.** Given an elliptic quartic $Q$ with a cusp and a node, $r_Q(-1) > 0$ if the node of $Q$ is inflectional iff $Q \sim C^*_{\lambda, \omega}$ with $(\lambda, \omega) \in \{(1, r), (r, r + 1), (r + 1, 1)\}$.
Proof. By Proposition 8.1 and Remark 8.2, we may assume that \( Q = C^*(a, a, a) \), with \( 2[a + \hat{a}] \neq 0 \neq 4[a], [a] \neq [\hat{a}] \neq -3[a] \). Let \( Q \xrightarrow{\nu} \hat{Q} \) be induced through \( \nu \) by \([t] \mapsto [n - t]\). Since \( g \) fixes \( q \), \([n] = [2]\). Since \( g \) must fix the branches of \( q \), \( 2[\hat{a}] = 2[a] \) and \( 8[a] = 0 \), i.e., \( \hat{q} \) is fixed. With \( \omega := 2(a - \hat{a}) \) and \( \lambda := 8a - \omega \), we get \( Q = C^*_{\lambda, \omega} \). By Remark 8.2, \( C^*_{\lambda, \omega} \sim C^*_{\omega, \lambda} \), and \( C^{*}_{\lambda+2\alpha, \omega} \sim C^*_{\lambda, \omega+2\alpha} \) for all \( \alpha \in \Omega_{\tau} \), hence we may assume \((\lambda, \omega) \in \{(1, \tau), (\tau, \tau + 1), (\tau + 1, 1)\}\). Note that \([t] \mapsto \frac{25}{2} - t\) induces through \( \nu \) an automorphism of \( C^*_{\lambda, \omega} \) that extends to \( \mathbb{P}^2 \).

**Lemma 8.8.** Let \( Q = C^*_{\lambda, \omega} \), with \( \lambda, \omega \), and \( a, \hat{b} = \frac{\exp(\frac{\eta(\omega)}{\eta(\lambda)})}{\exp(\frac{\eta(\omega)}{\eta(\lambda)})} \). Then \( \alpha(a) = \frac{1}{2} \eta(\lambda + \omega) \), and \( \alpha^2(b, \hat{b}) = \frac{\exp(\frac{\eta(\omega)}{\eta(\lambda)})}{\exp(\frac{\eta(\omega)}{\eta(\lambda)})} \).

**Proof.** By Remark 8.3, \( \alpha(a) = \zeta(\frac{\lambda}{\omega}) + \zeta(\frac{\omega}{\lambda}) = \frac{1}{2} \eta(\omega + \lambda) \), and \( \alpha(b, \hat{b}) = \frac{\exp(\frac{\eta(\omega)}{\eta(\lambda)})}{\exp(\frac{\eta(\omega)}{\eta(\lambda)})} \). Since \( \mathcal{P}(t) = \frac{2\omega}{\lambda} \) and \( \mathcal{P}(t + \frac{\lambda}{\omega}) = \frac{2\omega}{\lambda} \), we obtain \( \alpha^2(b, \hat{b}) = \frac{2\omega}{\lambda} \).

The three finite critical values of \( \mathcal{P} \) are \( e_1 := \mathcal{P}(\frac{\lambda}{\omega}) \), \( e_2 := \mathcal{P}(\frac{\omega}{\lambda}) \), and \( e_3 := -e_1 - e_2 = \mathcal{P}(\frac{\lambda + \omega}{\lambda}) \). Recall that \( (\mathcal{P}^\prime)^2 = -4 \prod_i (\mathcal{P} - e_i) \), hence \( (\mathcal{P}^\prime)^2 = -2 \prod_i (\mathcal{P} - e_{i-1})(\mathcal{P} - e_{i+1}) \). We get \( (\mathcal{P}^\prime)^2 = -2(e_2 - e_1)(e_2 - e_3) \), hence \( (\mathcal{P}^\prime)^2 = -2(e_2 - e_1)(e_2 - e_3) \).

Given \( r \in \mathbb{R} \), let \( |r| \) denote the largest integer less than or equal to \( r \).

**Proposition 8.9.** Let \( Q = C^*_{\lambda, \omega} \), as in Remark 8.7. Given \( m \in \mathbb{Z} \), \( R_Q(m) \neq \emptyset \) iff \( \left( \frac{\mathcal{P}(\frac{\lambda}{\omega}) + 2\mathcal{P}(\frac{\omega}{\lambda})}{\mathcal{P}(\frac{\lambda}{\omega}) + 2\mathcal{P}(\frac{\omega}{\lambda})} \right)^{\frac{m-1}{2}} = 1 \), in which case \( R_Q(m) = \{ [(m - 1)\alpha] \} \).

**Proof.** Fix \([n] \in R_Q(m)\). Since \([t] \mapsto [m - t]\) must fix \([a]\), we may assume that \( n = (m - 1)\alpha \), and then \( 4m[n] = 0 \). By Lemma 8.8, \( \alpha(a, \hat{b}) = 0 \).

When \( m \) is even, we have \([m\hat{a} - n] = [m\hat{b} - n] \), and Remark 3.4 implies that \( \alpha(m\hat{a} - n, m\hat{b} - n) = \exp(\frac{4m}{\lambda} \eta(\lambda + \omega) \eta(\lambda - \omega)) \). Using Lemma 8.8, we calculate \( \alpha_{m, n}(\alpha, \hat{b}) = \left( \frac{\mathcal{P}(\frac{\lambda}{\omega}) + 2\mathcal{P}(\frac{\omega}{\lambda})}{\mathcal{P}(\frac{\lambda}{\omega}) + 2\mathcal{P}(\frac{\omega}{\lambda})} \right)^{\frac{m-1}{2}} \).

When \( m \) is odd, we have \( m\hat{a} - n = \hat{a} - \omega \) and \( m\hat{b} - n = \hat{b} - \omega \), with \( p := \frac{m-1}{2} \). By Remark 3.4, \( \alpha(m\hat{a} - n, m\hat{b} - n) = (-1)^p \alpha(\alpha, \hat{b}) \exp(\frac{4p}{\lambda} \eta(\lambda + \omega) \eta(\lambda - \omega)) \).

By Lemma 8.8, \( \alpha_{m, n}(\alpha, \hat{b}) = \left( \frac{\mathcal{P}(\frac{\lambda}{\omega}) + 2\mathcal{P}(\frac{\omega}{\lambda})}{\mathcal{P}(\frac{\lambda}{\omega}) + 2\mathcal{P}(\frac{\omega}{\lambda})} \right)^{\frac{m-1}{2}} \).

**Corollary 8.10.** Up to Möbius equivalence, \( C^*_{\lambda, \omega} \) is the only elliptic quartic with a cusp and an inflectional node that admits a tangent process.

**Proof.** The condition \( \left( \frac{\mathcal{P}(\frac{\lambda}{\omega}) + 2\mathcal{P}(\frac{\omega}{\lambda})}{\mathcal{P}(\frac{\lambda}{\omega}) + 2\mathcal{P}(\frac{\omega}{\lambda})} \right)^2 = 1 \) means \( \mathcal{P}(\frac{\lambda}{\omega}) = 0 \). In this case, \( C^* \) is isomorphic to \( (y^2 - x^3)^2 \), i.e., \( \tau = 1 \). Since \( \mathcal{P}(\hat{u}) = -\mathcal{P}(t) \), we have \( \mathcal{P}(\frac{\lambda}{\omega}) = 0 \), hence \( \left[ \frac{\lambda}{\omega} \right] = \left[ \frac{\omega}{\lambda} \right] \). By Remark 8.7, we may assume \( (\lambda, \omega) = (1, 1) \).

**Example 8.11.** Let \( Q := (h \neq 0) \), with \( h := (x^2 - y^2)z^2 - xy^3 \). Then \( Q \sim C^*_{\lambda, \omega} \), with \( q \) a cusp and \( (0, 0, 1) \) and \( (1, 0, 0) \) of \( T_q(Q) \). The tangent process on \( Q \) maps \( p \) to the residual intersection \( g(p) \) of \( Q \) with the conic that passes through \( q \), \( n \), \( r \), \( p \), and is tangent to \( Q \) at \( p \). We calculate \( g(x, y, z) = [4(xy + z)^2 - (x^2 + y^2)^2, 16z^2 - 4xy(x^2 - y^2), 8z(x^2 + y^2)] \).
The group $\text{Aut}(Q)$ is generated by $[x, y, z] \mapsto [x, -y, iz]$. The regular extensions of $g$ that commute with $\text{Aut}(Q)$ are obtained by adding $ah$ to the second component of $g$, with $0 \neq a$. When $a = \frac{16b}{b-1|b+3|}$, with $|b^2 - 1| < 1$ and $|b^2 + 1| < 2$, the fixed points $\left[\pm \sqrt{\frac{b^2 - 1}{b+3}}, 1, 0\right]$ are attracting for the self-map $f_b$ obtained in this way; by Remark 5.5, $J(f_b)$ is the common boundary of their basins. Figure 4 shows the traces of $J(f_{-0.9})$ on the lines $(x = 0)$ (left) and $(z = 0)$ (right), near $[0, 1, 0]$.

8.4. Invariant cuspidal quartic. We discuss now the invariance of the elliptic quartics with two cusps.

Remark 8.12. Given an elliptic quartic $Q$ with two cusps, $Q \sim C(a) := C(a, a, -a)$, with $4[a] \neq 0$. Moreover, $\text{Aut}(Q) = \text{Aut}_Q(\mathbb{P}^2)$, and $r_Q(-1) = 1$. The tangents to $Q$ at the cusps meet at a flex of $Q$ iff $Q = C^{\rho}\left(\frac{1+a}{2}\right)$, with $\rho := \exp(\frac{\pi i}{2})$.

Proof. The first statement follows from Proposition 8.1 and Remark 8.2. Let $g \in \text{Aut}(Q)$, with $Q = C(a)$, be induced through $\nu$ by $C \xrightarrow{h} C$, $h[t] = [mt-n]$. Since $h$ leaves invariant the set $\{[a], [a]\}$, we get $2[n] = 0$. By Remark 8.2, $g \in \text{Aut}_Q(\mathbb{P}^2)$. When $m = -1$, we must have $h[a] = -[a]$, i.e. $[n] = 0$.

The tangents to $C(a)$ at the two cusps meet at $r := [0, 0, 1]$. Clearly, $r \in C(a)$ iff $6[a] = 0$. In this case, $r = \nu[3a]$, and $r$ is a flex of $C(a)$ iff $\left(\frac{\partial}{\partial x}\right)^3(3a) = 0$, i.e. $3(2a) = \eta(6a)$. By the addition theorems of $\zeta$ and $P$, this happens iff $P''(2a) = 0$ iff $P(2a) = 0$. Assuming this, the second-order differential equation of $P$ implies that $\sum e_{i-1}e_{i+1} = 0$, where $e_i$ are the three finite critical values of $P$. Therefore, $C$ is isomorphic to $C^{\rho}$, in which case $P(pt) = -\rho P(t)$ for all $t$, hence $P(\frac{1+a}{2}) = 0$. It follows that $2[a] = \pm \frac{1+a}{3}$, and we may assume $a = \frac{1-3}{2}$, by Remark 8.2.

Proposition 8.13. Let $Q = C(a)$, as in Remark 8.12. Given $m \in \mathbb{Z}$, $R_Q(m) \neq \emptyset$ iff there exists $e \in \{-1, 0, 1\}$ so that $2(m-e)[a] = 0$ and $(m-e)(2a) = \eta(2(m-e)a)$. When $e = 0$, $R_Q(m) = \{(m \pm 1)[a]\}$; when $e = \pm 1$, $R_Q(m) = \{(m-e)[a]\}$.

Proof. Given $n \in R_Q(m)$, let $Q \xrightarrow{\nu} Q$ be induced through $\nu$ by $[t] \mapsto [mt-n]$. Recall that $q = \nu[a]$ and $\bar{q} = \nu[-a]$ are the cusps of $Q$. There are three possibilities.
1. \(g(q) = q\) and \(g(\bar{q}) = \bar{q}\), i.e., \(2(m - 1)[a] = 0\) and \([n] = (m - 1)[a]\).
2. \(g(\bar{q}) = g(\bar{q})\), i.e., \(2m[a] = 0\) and \([n] = (m + 1)[a]\).
3. \(g(q) = q\) and \(g(\bar{q}) = q\), i.e., \(2(m + 1)[a] = 0\) and \([n] = (m + 1)[a]\).

Therefore, \(2(m - \epsilon)a = \omega \in \Omega\), for a (unique) \(\epsilon \in \{-1, 0, 1\}\). When \(\epsilon = \pm 1\), we may assume \(n = (m - \epsilon)a\). When \(\epsilon = 0\), we may assume \(n = (m - 1)a\) (conjugate if necessary \(g\) with the involution induced by \([t] \mapsto [-t]\)).

By Remark 8.3, \(\alpha(a) = 2\zeta(2a) = -\alpha(-a)\).

Assume \(\epsilon = \pm 1\). Then \(\alpha(ma - n) = \epsilon a(a), \alpha_{m,n}(a) = 2m(\eta(\omega) - (m - \epsilon)\zeta(2a))\), \(\alpha(-ma - n) = \alpha(-\epsilon a - \omega) = -\epsilon a(4\eta(\omega)), \alpha_{m,n}(-a) = -\alpha_{m,n}(a)\).

Assume \(\epsilon = 0\). Then \(\alpha(ma - n) = a(a), \alpha_{m,n}(a) = 2(m - 1)(\eta(\omega) - m\eta(2a))\), \(\alpha(-ma - n) = a(-a - \omega) = a(-a - 4\eta(\omega), \alpha_{m,n}(-a) = -2(m + 1)(\eta(\omega) - m\eta(2a))\).

**Corollary 8.14.** Up to Möbius equivalence, \(C^\omega(\frac{1+\omega}{6})\) is the only elliptic quartic with two cusps that admits a tangent process.

**Proof.** By Proposition 8.13, \(6[a] = 0\) and \(3\zeta(2a) = \eta(6a)\). (The proof of) Remark 8.12 finishes the proof.

**Example 8.15.** Let \(Q := (h = 0)\), with \(h := (y^2 - z^2)^2 - 8x^3z\). Then \(Q \sim C^\omega(\frac{1+\omega}{6})\); the tangents to \(Q\) at the cusps \(q := [0, 1, 1], \bar{q} := [0, -1, 1]\) meet at the flex \(r := [1, 0, 0]\). The tangent process on \(Q\) maps \(p \in Q\) to the residual intersection \(g(p)\) of \(Q\) with the conic that passes through \(q, \bar{q}, r, p\), and is tangent to \(Q\) at \(p\). We calculate \(g(x, y, z) = [x(x^3 - 2y^2z + 10z^3), y(7z^3 - 5y^2z - 2x^3), 2y^4]\).

The group \(\text{Aut}(Q)\) is generated by \([x, y, z] \mapsto [ax, y, -z]\). The extensions \(f_{a}\) of \(g\) that commute with \(\text{Aut}(Q)\) are obtained by adding \(ah\) to the last component of \(g\). When \(3.484 < a < 10\), \(f_{a}\) has three attracting points on the invariant line \((y = 0)\), and, by Remark 5.5, \(J(f_{a})\) is the common boundary of their basins. Figure 8.4 shows the traces of \(J(f_{a})\) on the lines \((z = 0.01y)\) (left) and \((z = 0)\) (right), near \(r\).

**References**


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