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ERRATA FOR "CUBIC POLYNOMIAL MAPS WITH PERIODIC CRITICAL ORBIT, PART II: ESCAPE REGIONS"

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ABSTRACT. In this note we fill in some essential details which were missing from our paper. In the case of an escape region $\mathcal{E}_h$ with non-trivial kneading sequence, we prove that the canonical parameter t can be expressed as a holomorphic function of the local parameter $\eta = a^{-1/\mu}$ (where a is the periodic critical point). Furthermore, we prove that for any escape region $\mathcal{E}_h$ of grid period $n \geq 2$, the winding number ν of $\mathcal{E}_h$ over the t-plane is greater or equal than the multiplicity μ of $\mathcal{E}_h$.

A result which can be stated as follows is claimed in §6 of the paper *Cubic Polynomial Maps with Periodic Critical Orbit, Part II: Escape Regions,* Conformal Geometry and Dynamics **14** (2010), 68–112 (referred to below as [BKM]).

Assertion A. *For any escape region $\mathcal{E}_h$, the residue $\oint dt/2\pi i$ at the ideal point ∞_h is zero. Furthermore, whenever the kneading sequence of $\mathcal{E}_h$ is non-trivial, the indefinite integral $t = \int dt$ can be expressed as a holomorphic function of the local parameter $\eta = \xi^{1/\mu} = a^{-1/\mu}$.*

This assertion is true; however, there is a gap in our proof when the kneading sequence is non-trivial. In this case, [BKM, Lemma 5.19 and Theorem 6.2] do show that the quotient dt/da can be expressed as a locally holomorphic function of η, vanishing at $\eta = 0$. However, this is not enough to prove the assertion.[1] Since $a = \eta^{-\mu}$, we have

$$\frac{dt}{d\eta} \;=\; \frac{dt}{da}\,\frac{da}{d\eta} \;=\; -\mu\,\frac{dt}{da}\,\eta^{-\mu-1}\,.$$

Thus we must show that dt/da is divisible by $\eta^{\mu+1}$ in order to complete the proof. In fact, we will prove a slightly sharper statement. The necessary details follow.

Lemma B. *Consider a Branner-Hubbard marked grid of period*[2] *$n \geq 2$, denoting its finite column heights by $L_1, \ldots, L_{n-1}$. If $L_{n-1} > 0$, then*

$$L_j \;=\; L_{n-1} - j \qquad \text{for} \qquad 1 \leq j \leq L_{n-1}\,.$$

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[1] Our mistake was to ignore the ξ^2 in the denominator of [BKM, Equation (6.3)].

[2] The period p of the critical orbit can be any multiple of the grid period n; but we will work only with the grid. Note that $n \geq 2$ if and only if the kneading sequence is non-trivial.

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Proof. Let $\{a_i\}$ be the periodic critical orbit. We will write the puzzle metric $d(a_i,\, a_j)$ of [BKM, Definition 3.7] briefly as $d(i,\, j)$, with $i, j \in \mathbb{Z}/n$, and with $d(0,\, i) = 2^{-L_i}$. The argument will be based on the following statement from [BKM, Lemma 3.8]. $\square$

Expanding property. *The equality*

$$d(i+1,\, j+1) \;=\; 2\, d(i,j)$$

holds provided that $d(i,j) < 1$, *and provided that* $\{0,\, i,\, j\}$ *do not form the vertices of an equilateral triangle in this metric.*

Using this, we will prove inductively that

$$(*_j) \qquad d(0,\, j) \;=\; d(j-1,\, j) \;=\; 2^{j-N}$$

for $1 \le j \le N$. To begin the induction, since the degenerate triangle with vertices $\{0,\, 0,\, n-1\}$ is certainly not equilateral, the equation $d(0,\, n-1) = 2^{-N} < 1$ implies that

$$d(1, n) \;=\; 2\, d(0,\, n-1) \;=\; 2^{1-N}\,.$$

Since $d(1,\, n) = d(0,\, 1)$, this proves Equation $(*_1)$. Now suppose inductively that $(*_j)$ holds for $j < k$, where $2 \le k \le N$. Then the triangle $\{0,\, k-2,\, k-1\}$ is not equilateral, hence

$$d(k-1,\, k) \;=\; 2\, d(k-2,\, k-1) = \; 2^{k-N}\,.$$

Together with the induction hypothesis, this proves that $d(0,\, k-1) < d(k-1,\, k)$. Therefore the ultrametric property (the statement that the two longest edges of any triangle must have equal length) implies that $d(0,\, k) = d(k-1,\, k)$. This completes the induction. Since $d(0, j) = 2^{-L_j}$, we have also proved that $L_j = N - j$, as required. $\square$

It will be convenient to use the abbreviated notation $A_\ell(j)$ for the Branner-Hubbard annulus $A_\ell(a_j)$. As in the proof of [BKM, Lemma 5.19], let[3]

$$\mathfrak{S}_j \;=\; \sum_{\ell=0}^{\infty} \mathrm{MOD}(A_\ell(j))$$

be the sum of all of the moduli for the j-th column, normalized so that $\mathrm{MOD}(A_0(j)) = 2$.

Lemma C. *The inequality*

$$\mathfrak{S}_1 \;\ge\; \mathfrak{S}_n + 2 \;=\; \mathfrak{S}_0 + 2$$

holds whenever the grid period satisfies $n \ge 2$, *with strict inequality when* $n > 2$.

Proof. As in the proof of the weaker inequality $\mathfrak{S}_1 > \mathfrak{S}_n$ following the statement of [BKM, Lemma 5.19], the idea is to note that each critical modulus $\mathrm{MOD}(A_\ell(n))$ is equal to some $\mathrm{MOD}(A_{\ell'}(1))$ from the first column, where the correspondence $\ell \mapsto \ell' = \ell'(\ell) \ge \ell$ is strictly monotone, with $\ell' = \ell + n - 1$ for large ℓ.

[3] As an example, in Figures 1 and 2, the moduli for the points in the zero-th column at depth $0 \le \ell \le 7$ can be computed from [BKM, Lemma 5.7] as $2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8}$, with $\mathrm{MOD}(A_\ell(0)) = \mathrm{MOD}(A_{\ell-5}(0))/2$ for $\ell > 7$. The sum is $\mathfrak{S}_0 = \frac{31}{4} = 7\frac{3}{4}$.

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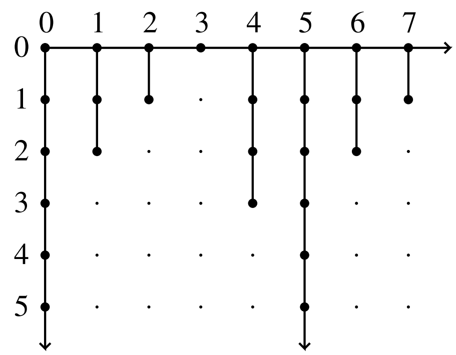

FIGURE 1. *Sample grid of period $n = 5$. Here the column heights are $L_0 = \infty\,,\ L_1 = 2\,,\ L_2 = 1\,,\ L_3 = 0\,,\ L_4 = 3\,,\ \ldots$.*

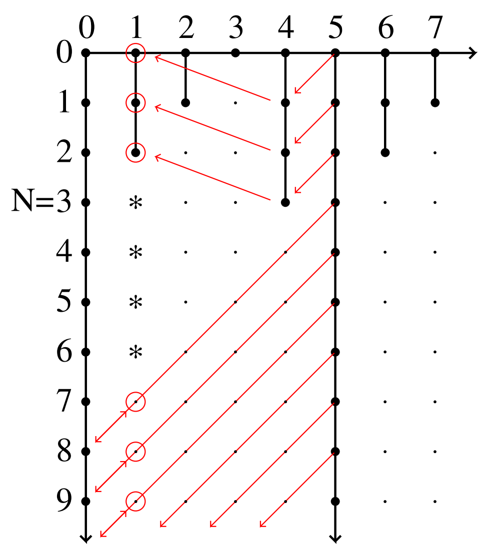

FIGURE 2. *The correspondence $\ell \mapsto \ell'$.*

This correspondence can be described as follows. Start with the marked grid point in the n-th column at depth ℓ and follow the south-west diagonal until hitting another marked point, say in column $n - \delta$ at depth $\ell + \delta$. Then by definition

$$\ell'(\ell) \;=\; \ell + \delta - 1\,,$$

one level higher than the hitting point. (Compare Figure 2, where each grid point of level ℓ' in the first column is circled.) Using [BKM, Lemma 5.7], it is a straightforward exercise to prove that $\mathrm{MOD}(A_\ell(n))$ is equal to $\mathrm{MOD}(A_{\ell'}(1))$. (Both are equal to $2\,\mathrm{MOD}(A_{\ell'+1}(0))$.)

Evidently, there must be exactly $n-1$ levels which do not lie in the image of this correspondence $\ell \mapsto \ell'$. The corresponding points in the first column are indicated by asterisks in Figure 2. Thus the difference $\mathfrak{S}_1 - \mathfrak{S}_n$ is precisely equal to the sum of the $n - 1$ moduli $\mathrm{MOD}(A_\ell(1))$ associated with these asterisk points. Setting $N = L_{n-1} \geq 0$, it is easy to check that $\ell' = \ell$ for $\ell < N$; but that $\ell' > \ell$ when

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$\ell = N$. Thus the grid point at depth N in column one will always be the highest asterisk point. Since it follows easily from Lemma B that $\mathrm{MOD}(A_N(1)) = 2$, this proves Lemma C. $\square$

Proof of Assertion A. Setting $\delta = \mathfrak{S}_1 - \mathfrak{S}_n \geq 2$, the proof of [BKM, Lemma 5.19 and Theorem 6.2] show that dt/da can be expressed as $\xi^\delta = \eta^{\delta\mu}$ multiplied by a function of η which is holomorphic near the ideal point. Hence $dt/d\eta$ is equal to $\eta^{(\delta-1)\mu-1}$ multiplied by a locally holomorphic function. Since $\delta \geq 2$ and $\mu \geq 1$, we have $(\delta - 1)\mu - 1 \geq 0$. Therefore $dt/d\eta$ is locally holomorphic, which implies that the indefinite integral t is locally holomorphic, as required. $\square$

In fact this argument proves a slightly stronger result. Choosing the additive constant so that t vanishes at the ideal point, we see that t is equal to $\eta^{(\delta-1)\mu} = \xi^{\delta-1}$ times a locally holomorphic function, where $\delta \geq 2$ with strict inequality when $n > 2$. Setting

$$t \;=\; \beta\,\xi^{\nu/\mu} + \text{(higher order terms)} \qquad \text{with} \qquad \beta \in \mathbb{C}\,,\ \beta \neq 0\,,$$

we obtain the following.

Assertion D. *For any escape region of grid period $n \geq 2$, the winding number ν and the multiplicity $\mu \geq 1$ are related by the inequality $\nu \geq \mu$, with strict inequality when $n > 2$.*

References

[BKM] A. Bonifant, J. Kiwi and J. Milnor, *Cubic Polynomial Maps with Periodic Critical Orbit, Part II: Escape Regions,* Conformal Geometry and Dynamics **14** (2010) 68–112.

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