ERRATA FOR “CUBIC POLYNOMIAL MAPS WITH PERIODIC CRITICAL ORBIT, PART II: ESCAPE REGIONS”

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Abstract. In this note we fill in some essential details which were missing from our paper. In the case of an escape region $E_h$ with non-trivial kneading sequence, we prove that the canonical parameter $t$ can be expressed as a holomorphic function of the local parameter $\eta = a^{-1/\mu}$ (where $a$ is the periodic critical point). Furthermore, we prove that for any escape region $E_h$ of grid period $n \geq 2$, the winding number $\nu$ of $E_h$ over the $t$-plane is greater or equal than the multiplicity $\mu$ of $E_h$.

A result which can be stated as follows is claimed in §6 of the paper Cubic Polynomial Maps with Periodic Critical Orbit, Part II: Escape Regions, Conformal Geometry and Dynamics 14 (2010), 68–112 (referred to below as [BKM]).

Assertion A. For any escape region $E_h$, the residue $\int dt/2\pi i$ at the ideal point $\infty_h$ is zero. Furthermore, whenever the kneading sequence of $E_h$ is non-trivial, the indefinite integral $t = \int dt$ can be expressed as a holomorphic function of the local parameter $\eta = \xi^{1/\mu} = a^{-1/\mu}$.

This assertion is true; however, there is a gap in our proof when the kneading sequence is non-trivial. In this case, [BKM] Lemma 5.19 and Theorem 6.2 do show that the quotient $dt/da$ can be expressed as a locally holomorphic function of $\eta$, vanishing at $\eta = 0$. However, this is not enough to prove the assertion. Since $a = \eta^{-\mu}$, we have

$$\frac{dt}{d\eta} = \frac{dt}{da} \frac{da}{d\eta} = -\mu \frac{dt}{da} \eta^{-\mu-1}.$$ 

Thus we must show that $dt/da$ is divisible by $\eta^{\mu+1}$ in order to complete the proof. In fact, we will prove a slightly sharper statement. The necessary details follow.

Lemma B. Consider a Branner-Hubbard marked grid of period $n \geq 2$, denoting its finite column heights by $L_1, \ldots, L_{n-1}$. If $L_{n-1} > 0$, then

$$L_j = L_{n-1} - j \quad \text{for} \quad 1 \leq j \leq L_{n-1}.$$ 

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1Our mistake was to ignore the $\xi^2$ in the denominator of [BKM] Equation (6.3).

2The period $p$ of the critical orbit can be any multiple of the grid period $n$; but we will work only with the grid. Note that $n \geq 2$ if and only if the kneading sequence is non-trivial.

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Proof. Let \( \{a_i\} \) be the periodic critical orbit. We will write the puzzle metric \( d(a_i, a_j) \) of [BKM, Definition 3.7] briefly as \( d(i, j) \), with \( i, j \in \mathbb{Z}/n \), and with \( d(0, i) = 2^{-L_i} \). The argument will be based on the following statement from [BKM, Lemma 3.8].

\[ \Box \]

**Expanding property.** The equality
\[
d(i + 1, j + 1) = 2 \, d(i, j)
\]
holds provided that \( d(i, j) < 1 \), and provided that \( \{0, i, j\} \) do not form the vertices of an equilateral triangle in this metric.

Using this, we will prove inductively that
\[
\text{(s)} \\
d(0, j) = d(j - 1, j) = 2^{j - N}
\]
for \( 1 \leq j \leq N \). To begin the induction, since the degenerate triangle with vertices \( \{0, 0, n - 1\} \) is certainly not equilateral, the equation \( d(0, n - 1) = 2^{-N} < 1 \) implies that
\[
d(1, n) = 2 \, d(0, n - 1) = 2^{1 - N}.
\]
Since \( d(1, n) = d(0, 1) \), this proves Equation (s1). Now suppose inductively that (s) holds for \( j < k \), where \( 2 \leq k \leq N \). Then the triangle \( \{0, k - 2, k - 1\} \) is not equilateral, hence
\[
d(k - 1, k) = 2 \, d(k - 2, k - 1) = 2^{k - N}.
\]
Together with the induction hypothesis, this proves that \( d(0, k - 1) < d(k - 1, k) \). Therefore the ultrametric property (the statement that the two longest edges of any triangle must have equal length) implies that \( d(0, k) = d(k - 1, k) \). This completes the induction. Since \( d(0, j) = 2^{-L_j} \), we have also proved that \( L_j = N - j \), as required.\[ \Box \]

It will be convenient to use the abbreviated notation \( A_\ell(j) \) for the Branner-Hubbard annulus \( A_\ell(a_j) \). As in the proof of [BKM, Lemma 5.19], let \( \Box \)
\[
\mathcal{E}_j = \sum_{\ell=0}^{\infty} \text{MOD}(A_\ell(j))
\]
be the sum of all of the moduli for the \( j \)-th column, normalized so that \( \text{MOD}(A_0(j)) = 2 \).

**Lemma C.** The inequality
\[
\mathcal{E}_1 \geq \mathcal{E}_n + 2 = \mathcal{E}_0 + 2
\]
holds whenever the grid period satisfies \( n \geq 2 \), with strict inequality when \( n > 2 \).

**Proof.** As in the proof of the weaker inequality \( \mathcal{E}_1 > \mathcal{E}_n \) following the statement of [BKM, Lemma 5.19], the idea is to note that each critical modulus \( \text{MOD}(A_\ell(n)) \) is equal to some \( \text{MOD}(A_{\ell'}(1)) \) from the first column, where the correspondence \( \ell \mapsto \ell' = \ell'(\ell) \geq \ell \) is strictly monotone, with \( \ell' = \ell + n - 1 \) for large \( \ell \).

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\[ \Box \]

As an example, in Figures [4] and [2] the moduli for the points in the zero-th column at depth \( 0 \leq \ell \leq 7 \) can be computed from [BKM, Lemma 5.7] as \( 2, 1, \frac{1}{2}, \frac{1}{4}, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \), with \( \text{MOD}(A_\ell(0)) = \text{MOD}(A_{\ell-5}(0))/2 \) for \( \ell > 7 \). The sum is \( \mathcal{E}_0 = \frac{31}{4} = 7 \frac{3}{4} \).
Figure 1. Sample grid of period $n = 5$. Here the column heights are $L_0 = \infty$, $L_1 = 2$, $L_2 = 1$, $L_3 = 0$, $L_4 = 3$, . . .

Figure 2. The correspondence $\ell \mapsto \ell'$. This correspondence can be described as follows. Start with the marked grid point in the $n$-th column at depth $\ell$ and follow the south-west diagonal until hitting another marked point, say in column $n - \delta$ at depth $\ell + \delta$. Then by definition $\ell'(\ell) = \ell + \delta - 1$, one level higher than the hitting point. (Compare Figure 2 where each grid point of level $\ell'$ in the first column is circled.) Using [BKM Lemma 5.7], it is a straightforward exercise to prove that $\text{MOD}(A_\ell(n))$ is equal to $\text{MOD}(A_{\ell'}(1))$. (Both are equal to $2 \text{ MOD}(A_{\ell'}(1))$.)

Evidently, there must be exactly $n - 1$ levels which do not lie in the image of this correspondence $\ell \mapsto \ell'$. The corresponding points in the first column are indicated by asterisks in Figure 2. Thus the difference $S_1 - S_n$ is precisely equal to the sum of the $n - 1$ moduli $\text{MOD}(A_\ell(1))$ associated with these asterisk points. Setting $N = L_{n-1} \geq 0$, it is easy to check that $\ell' = \ell$ for $\ell < N$; but that $\ell' > \ell$ when
\[ \ell = N. \] Thus the grid point at depth \( N \) in column one will always be the highest asterisk point. Since it follows easily from Lemma B that \( \text{MOD}(A_N(1)) = 2 \), this proves Lemma C.

**Proof of Assertion A.** Setting \( \delta = S_1 - S_n \geq 2 \), the proof of \([BKM]\) Lemma 5.19 and Theorem 6.2 show that \( dt/da \) can be expressed as \( \xi^\delta \eta^{\delta \mu} \) multiplied by a function of \( \eta \) which is holomorphic near the ideal point. Hence \( dt/d\eta \) is equal to \( \eta^{(\delta - 1)\mu - 1} \) multiplied by a locally holomorphic function. Since \( \delta \geq 2 \) and \( \mu \geq 1 \), we have \( (\delta - 1)\mu - 1 \geq 0 \). Therefore \( dt/d\eta \) is locally holomorphic, which implies that the indefinite integral \( t \) is locally holomorphic, as required.

In fact this argument proves a slightly stronger result. Choosing the additive constant so that \( t \) vanishes at the ideal point, we see that \( t \) is equal to \( \eta^{(\delta - 1)\mu} = \xi^{\delta - 1} \) times a locally holomorphic function, where \( \delta \geq 2 \) with strict inequality when \( n > 2 \). Setting

\[
t = \beta \xi^{\nu/\mu} + \text{(higher order terms)} \quad \text{with} \quad \beta \in \mathbb{C}, \ \beta \neq 0,
\]

we obtain the following.

**Assertion D.** For any escape region of grid period \( n \geq 2 \), the winding number \( \nu \) and the multiplicity \( \mu \geq 1 \) are related by the inequality \( \nu \geq \mu \), with strict inequality when \( n > 2 \).

**References**