Project II
Leslie Matrix or Population Projection Matrix
Due December 10, 2004

You can work in teams of 2 or 3 people.

(1) **The Leslie growth model** is a model for growth of a population that is naturally segmented into age classes. The relevant data for each age class are: the reproduction rate and the rate of survival into the next age class.

The following table lists reproduction and survivor rates for the female population of a certain species of domestic sheep in New Zealand. (For animal populations, it is conventional to consider only the females, since only they reproduce, and they are usually a fixed percentage of the total population.) Sheep give birth only once a year, which dictates a natural time step of one year. In the species under consideration, sheep seldom if ever live longer than 12 years, which gives a natural stopping point for the age class.

Birth and Survival Rates for Female New Zealand Sheep

age(years)	birth rate	survival rate
0-1	0.000	0.845
1-2	0.045	0.975
2-3	0.391	0.965
3-4	0.472	0.950
4-5	0.484	0.926
5-6	0.546	0.895
6-7	0.543	0.850
7-8	0.502	0.786
8-9	0.468	0.691
9-10	0.459	0.561
10-11	0.433	0.370
11-12	0.421	0.000

In any given year, a particular population (e.g., a single herd) can be represented by a *state vector* $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, where x_i represents the number of female sheep in the i-th age class. If absolute numbers are not known, a *state* may be represented by a vector of fractions of the population in each age class., i.e., by a vector whose entries sum to 1.

The Leslie growth matrix for the population is the transition matrix L from the state in one year to the state in the next year. Thus, if $\vec{x}$ is the state vector in a given year, the state vector after one year's growth is $L\vec{x}$ and the growth in that year (distributed in age classes) is $L\vec{x} - \vec{x}$.

In one year's time, only two types of transitions are possible:

(a) A sheep gives birth, adding one to the youngest age class. Thus, the rates in the birth column are entries of the first row of L.
(b) A sheep survives the year to enter the next age class. Thus, the survival rates are entries of the form $L_{j+1,j}$, that is the entries just below the main diagonal of L.

All other entries of L are zero.

(2) **Activities:**
  (a) Enter the Leslie matrix L for the table given above.
  (b) Use maple to find the characteristic polynomial of the Leslie matrix.
  (c) Plot the characteristic polynomial of the Leslie matrix. What can you say from the plot about the eigenvalues of the characteristic polynomial of L.?
  (d) Compute the eigenvalues of L directly. How many real eigenvalues are there? How can you explain this using the plot of the characteristic polynomial?
  (e) Find the eigenvectors for the positive eigenvalues. Modify your eigenvectors (if necessary) to make it a state vector, and describe what it tells you about a distribution into age classes. (**Hint:** Any multiple of an eigenvector is also an eigenvector. Your *state* vector should have entries which sum to 1.)
  (f) If the state vector you found in the previous step is the age distribution in a given year, what will the distribution be in one year? In ten years?

(3) **Theorems about Leslie Matrices**
  (a) A Leslie matrix L has a unique positive eigenvalue λ_1. This eigenvalue has multiplicity one, and it has an eigenvector $\vec{x}_1$ whose entries are all positive.
  (b) If λ_1 is the unique positive eigenvalue of L, and λ_i is any other eigenvalue (real or complex), then $|\lambda_i| \leq \lambda_1$. That is λ_1 is a dominant eigenvalue.
  (c) If any two successive entries a_j and a_{j+1} of the first row of L are both positive, then $|\lambda_i| < \lambda_1$ for every other eigenvalue. That is the females in two successive age classes are fertile (almost always the case in any realistic population) then λ_1 is a strictly dominant eigenvalue.
  (d) Let x^k denote the state vector $L^k x^0$ after k growth periods. If λ_1 is a strictly dominant eigenvalue, then for large values of k, x^{k+1} is approximately $\lambda_1 x^k$, no matter what the starting state x^0. That is, as k becomes large, successive state vectors become more and more like an eigenvector for λ_1.

Remark. We saw in the example of the New Zealand sheep population an illustration of the first three theorems about Leslie matrices.

Theorem (d) needs careful interpretation. It doesn't say that the sequence of states converges –in particular, if the dominant eigenvalue is > 1, the sequence does not converge at all. On the other hand, if we "normalize" the state vector at each step – say, by making its entries sum to 1– the sequence of modified state vectors does converge to an eigenvector. Normalized or not, the sequence shows us an equilibrium age distribution of the female population, which is approached over time.

(4) **Activities**

(a) For an arbitrary starting vector x_0, compute $x_k = L^k x_0$ for a modest value of k. Then compute Lx_k and $\lambda_1 x_k$ and explain the meaning of each of these vectors. How do Lx_k and $\lambda_1 x_k$ compare?

(b) Vary k in the preceding step to find an eigenvector for λ_1. Modify your eigenvector (if necessary) to make it a *state vector*. (**Hint:** Your *state vector* will be more meaningful if its entries sum to 1) Describe what your state vector tells you about a distribution into age classes. How this state vector compares with the one that you obtained before?