MTH 436 Homework 4 – Solutions

1) Skipped. Will be assigned as a problem in Homework 8.

2) Suppose that f is continuous on $[a,b]$ and that $\int_a^b f = 0$

a) Prove that there exists a point $c \in [a,b]$ such that $f(c) = 0$.

b) Suppose that f is non-negative on $[a,b]$. Prove that $f(x) = 0$ for all x in $[a,b]$.

a) Proof: Since f is continuous on $[a,b]$, $F(x) = \int_a^x f$, is differentiable on $[a,b]$, and $F'(x) = f(x)$ for all $x \in [a,b]$. Thus by the mean value theorem there exists a point $c \in (a,b)$ such that $F'(c) = \frac{(F(b)-F(a))}{b-a}$. Once again using the fundamental theorem of calculus, $\int_a^b f = F(b) - F(a) = 0$. Thus there exists a point $c \in [a,b]$ such that $f(c) = F'(c) = \frac{(F(b)-F(a))}{b-a} = \frac{\int_a^b f}{b-a} = 0$. ∎

b) Proof: Since f is non-negative on $[a,b]$, $F(x) = \int_a^x f$, is increasing on $[a,b]$. Thus given $x \in [a,b]$, $F(a) \le F(x) \le F(b)$. However $F(a) = 0$ by definition, and $F(b) = \int_a^b f = 0$. Thus $F(x) = 0$ for all $x \in [a,b]$. Thus $F'(x) = 0$ for all $x \in [a,b]$. Thus $f(x) = 0$ for all $x \in [a,b]$. ∎

A proof without the FTC is also possible. Prove (b) first. Suppose that for some $x_0 \in [a,b]$, $f(x_0) > 0$. As f is continuous at x_0, there exist $\eta > 0$ and $\delta > 0$ such that $f(x) \geq \eta$ for all $x \in [x_0-\delta, x_0+\delta]$. Hence;

$$\int_a^b f = \int_a^{x_0-\delta} f + \int_{x_0-\delta}^{x_0+\delta} f + \int_{x_0+\delta}^{b} f \geq 0 + \eta \cdot 2\delta + 0 > 0.$$

(If $x_0 = a$ or $x_0 = b$, we consider $[x_0-\delta, b]$ or $[a, x_0+\delta]$ instead of $[x_0-\delta, x_0+\delta]$.) Contradiction as $\int_a^b f = 0$. Similarly, we obtain a contradiction if $f(x_0) < 0$ and (b) is proved.

Similarly, we prove that if f is nonpositive, i.e. $f(x) \le 0$ for all $x \in [a,b]$, then $f(x) = 0$ for all $x \in [a,b]$.

(a) follows from (b). Suppose there is no $c \in [a,b]$ such that $f(c) = 0$. As f is continuous, ~~by~~ the IVT implies that $f(x) > 0$ for $x \in [a,b]$ or $f(x) < 0$ for all $x \in [a,b]$. By (b), $f(x) = 0$ for all $x \in [a,b]$. A contradiction. Hence, (a).

3) Straightforward from Problem 2. Consider $h = f - g$. Then h is continuous on $[a,b]$ and $\int_a^b h = 0$. By (a) of Problem 2, there exists $c \in [a,b]$ such that $h(c) = f(c) - g(c) = 0$. Hence, $f(c) = g(c)$.

4) Find the derivative of the function F defined by

$$F(x) = \int_0^{x^2} t^2 \sin t^2 \, dt.$$

proof: Let $G(x) = \int_0^x t^2 \sin t^2 \, dt$ and $g(x) = x^2$.

Thus,

$$F(x) = G(g(x)).$$

Hence, by Theorem 32.3 (Chain Rule)

$$F'(x) = G'(g(x)) \cdot g'(x) = (t^2 \sin t^2)\big|_{t=x^2} \cdot 2x =$$

$$= x^4 \sin x^4 \, (2x)$$

$$= 2x^5 \sin x^4$$

5) Take an interval $[a,b] \subseteq I$ such that $c \in (a,b)$. Consider the running integral

$$F(x) = \int_a^x f \quad \text{for } x \in [a,b]. \qquad (1)$$

Since f is continuous at c, $F'(c) = f(c)$ by Th 34.2 (The 2nd FTC.). From the definition of the derivative, (1), and Cor 33.1:

$$F'(c) = f(c) = \lim_{h \to 0} \frac{F(c+h) - F(c)}{h} = \lim_{h \to 0} \frac{\int_c^{c+h} f}{h}.$$

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