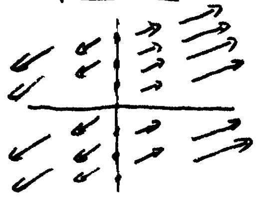
MTH243 Sec 2,3 Quiz 8 - Solutions

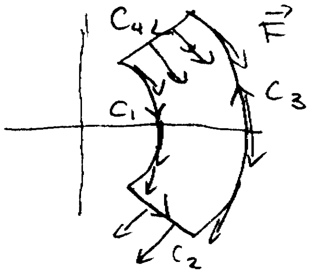
#6 p.803: $\vec{F} = 2x\vec{i} + x\vec{j} = x(2\vec{i}+\vec{j}) \parallel 2\vec{i}+\vec{j}$. The vector field is parallel to the vector $2\vec{i}+\vec{j}$, pointing the same direction for $x>0$, the opposite direction for $x<0$. Magnitude increases as $|x|$ increases.



#12 p.803: $\vec{F} = -y\vec{i}$

#16. p.833: $\vec{F} = y\vec{i} - x\vec{j}$ is perpendicular to $x\vec{i}+y\vec{j}$, hence it is tangent to each circle centered at the origin pointing CW. Thus, $\vec{F}$ is perpendicular all the time to C_4 and C_2 and

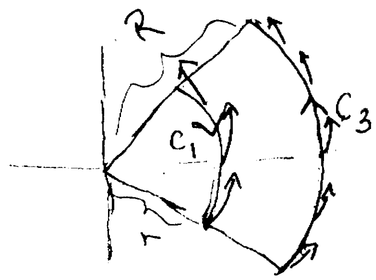
$$\int_{C_4} \vec{F} = \int_{C_2} \vec{F} = 0.$$



$\int_{C_3} \vec{F} < 0$ and $\int_{C_1} \vec{F} > 0$ and $\int_{C_3} \vec{F}$ is larger in magnitude as $\vec{F}$ is larger in magnitude along C_3 and C_3 is longer than C_1. Thus, $\int_C \vec{F} < 0$.

#18 p.833: $\vec{F}(x,y) = -\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$ is parallel to the vector field from the previous example, hence

$$\int_{C_4} \vec{F} = \int_{C_2} \vec{F} = 0.$$



We shall show that

$$\int_{C_3} \vec{F} = -\int_{C_1} \vec{F}. \quad (1)$$

Hence, $\int_C \vec{F} = 0$. To show (1), denote by r and R the radius of C_1 and C_3 respectively. Observe that

$$\|\vec{F}(x,y)\| = \frac{1}{\sqrt{x^2+y^2}}.$$

Thus, $\|\vec{F}(x,y)\|$ along C_1 is $\frac{1}{r}$, and along C_3 it is $\frac{1}{R}$. $\vec{F}$ is tangent to C_1 and C_3 (So $\vec{F}$ is shorter along C_3 and longer along C_1). Let's look at Riemann sums for $\int_{C_1}\vec{F}$ and $\int_{C_3}\vec{F}$. For $\int_{C_1}\vec{F}$ a Riemann sum is:

$$\sum_i \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i \approx \sum_i -\|\vec{F}(\vec{r}_i)\|\cdot\|\Delta\vec{r}_i\| = -\frac{1}{r}\sum_i \|\Delta\vec{r}_i\| \approx -\frac{1}{r}(\text{length of } C_1).$$

because $\vec{F}$ is tangent to C_1 pointing the opposite direction

Thus, we are estimating that

$$\int_{C_1} \vec{F} = -\frac{1}{r} \cdot \text{Length } C_1$$

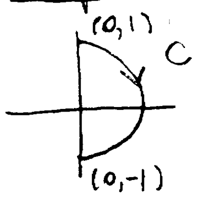
Similarly, we estimate that

$$\int_{C_3} \vec{F} = \frac{1}{R} \cdot \text{Length } C_3$$

By similarity, $\frac{1}{r} \cdot \text{Length } C_1 = \frac{1}{R} \text{Length } C_3$, thus $\int_{C_1} \vec{F} = -\int_{C_3} \vec{F}$.

The above reasoning is not very precise but this is the best you could do in section 18.1. You could show $\int_C \vec{F} = 0$ precisely by parametrizing C_1, C_2, C_3, C_4. for fixed r and R.

<u>#6 p 838</u>:



$\vec{F} = y\vec{i} - x\vec{j}$.

C: $x = \sin t$, $y = \cos t$, t in $[0, \pi]$, or

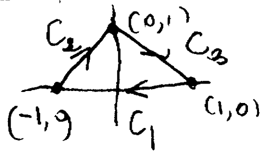
$\vec{r}(t) = \sin t\,\vec{i} + \cos t\,\vec{j}$.

$\vec{r}\,'(t) = \cos t\,\vec{i} - \sin t\,\vec{j}$, $\vec{F}(\vec{r}(t)) = \cos t\,\vec{i} - \sin t\,\vec{j}$.

Hence:

$$\int_C \vec{F}(\vec{r})d\vec{r} = \int_0^{\pi} \vec{F}(\vec{r}(t))\vec{r}\,'(t)dt = \int_0^{\pi} (\cos^2 t + \sin^2 t)dt = \pi.$$

<u>#9 p. 839</u>: $\vec{F} = xy\vec{i} + (x-y)\vec{j}$, C is the path on the picture:



$$\int_C \vec{F} = \int_{C_1} \vec{F} + \int_{C_2} \vec{F} + \int_{C_3} \vec{F}$$

Along C_1; $y = 0$, so $\vec{F} = x\vec{j}$ is perpendicular to C_1. Thus $\int_{C_1} \vec{F} = 0$.

We have to parametrize C_2 and C_3.

C_2: $\vec{r}(t) = (-1+t)\vec{i} + t\vec{j}$, t in $[0,1]$

So $$\int_{C_2} \vec{F} = \int_0^1 (t(-1+t)\vec{i} - \vec{j}) \cdot (\vec{i} + \vec{j})\, dt = -\frac{7}{6}.$$

Similarly, by parametrizing C_3:

$$\int_{C_3} \vec{F} = \frac{1}{6}$$

Thus, $\int_C \vec{F} = -1$. The integral is easier to find using Green's Theorem. Try it.