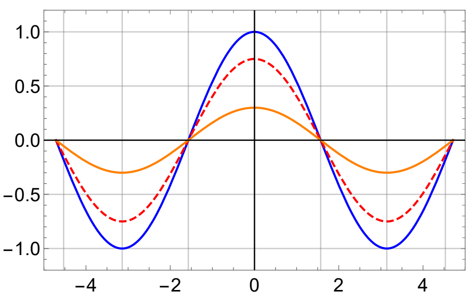
APPLIED PRECALCULUS



Erin Denette and Barbara Kaskosz

Department of Mathematics and Applied Mathematical Sciences
University of Rhode Island

Preface

As the title of the book suggests, our emphasis is on the real-life meaning of precalculus concepts. We want our students to master the basic concepts of precalculus while building a solid understanding of those concepts in practical contexts and the skills to apply mathematical tools to real-life processes.

To illustrate our approach, take for example the family of linear functions. When talking about linear functions, our emphasis is on the practical meaning and the units of the slope as well as on the units and the practical meaning of the horizontal and vertical intercepts. Such practical understanding is built through many applied examples and their careful analysis. We show how to use linear functions to model real-life processes which are described through tabulated data. At the same time, we try to ensure that students build the algebraic proficiency necessary to successfully apply linear functions.

We present all the main precalculus topics while stressing their practical meaning and their real-life applications. Those topics include:

- Functions, their graphs, and their numerical representation;
- Linear functions and their applications;
- Quadratic functions and quadratic equations;
- Algebra of exponential expressions, power functions, and their applications;
- Exponential and logarithmic functions, their algebraic properties and their rich variety of applications in the life sciences;
- Trigonometric functions and their applications to modeling periodic phenomena.

Our intended audience is primarily students in the life sciences and pharmacy. These students can be divided generally into two groups. The first group consists of students for whom this course is a terminal mathematics course, who wish to learn tools of before-calculus mathematics and to develop solid skills of applying those tools to real-life problems. The second group consists of students who plan to continue their study of mathematics and intend to tackle next an applied calculus course.

About the Authors

Erin Denette serves as a Senior Lecturer for the University of Rhode Island Department of Mathematics. She received her Ph.D. in Mathematics from the University of Rhode Island. She earned her M.S. from the University of Rhode Island and her B.S. from Fitchburg State University.

Barbara Kaskosz serves as a Professor for the University of Rhode Island Department of Mathematics. She received her Ph.D. in Mathematics from the Institute of Mathematics of the Polish Academy of Sciences and her M.S. in Mathematics from the University of Warsaw.

Contents

Chapter 1

Introduction to Functions

1.1 The Concept of a Function

One of the main concepts of mathematics and one of the main concepts used to model applied processes is the concept of a function. A mathematical definition of a function is abstract and general but in the context of real-life applications that we will study a function is a special kind of dependence between two numerical variables. Let's look at an example.

Example 1.1.1. You are buying a gift from the Ross-Simons online jewelry store. The amount of money, C, that you will have to spend, the cost, depends on the list price, p, of the item you select. The cost C is equal to the price p plus the 7% sales tax plus \$4.95 for shipping and handling. In other words, given the list price p, you can calculate the cost C, in dollars, as follows:

$$C = p + 0.07p + 4.95.$$

Note that $0.07p$ is 7% of the price — the sales tax.

In this example, C and p are variables — their values *vary* based on the item you select. They are both *numerical* variables as their possible values are numbers. The cost C depends on p and moreover the cost is uniquely determined by the price p. We say in this case that C "is a function of p". Using the *functional notation*, we write:

$$C = f(p)$$

$C = f(p)$ is read as C equals "f of p". f does not stand for the word "function". Any other letter could be used instead. f is the name of the function and $f(p)$ is the *value* of the function f for a given list price p which is the cost corresponding to a list price p:

$$f(p) = p + 0.07p + 4.95.$$

For every p, the value of the function $f(p)$ gives the cost C for a list price p. For example, $f(50)$ — the value of the function at $p = 50$ — gives the cost corresponding to the list price 50 dollars, $f(60)$ gives the cost corresponding to the list price 60 dollars, and so on.

Question: How much money will you spend if you select a piece of jewelry listed at the list price $p = 80$ dollars?

We are looking for the cost C when $p = 80$. In other words, we are looking for $f(80)$ — "f of 80". Using the formula for $f(p)$ with $p = 80$, we obtain:

$$f(80) = 80 + 0.07 \cdot 80 + 4.95 = 90.55$$

If you select an item listed at p=80 dollars, you will pay $C = 90.55$ dollars. In functional notation: $90.55 = f(80)$ or equivalently $f(80) = 90.55$.

In Example 1.1.1, the cost variable, C, is what we call the *dependent variable* or the *output variable*. The list price, p, is the *independent variable* or the *input variable*. For each list price, for each input value, or input for short, we have exactly one cost value. So we have exactly one output value, or output for short, for each input. The formula for the function f gives the rule of how to obtain the output corresponding to each possible input p.

In general, we define a function as follows.

Informal Definition of a Function

A function is a correspondence between two numerical variables, the input variable and the output variable, which has the following properties: It takes as inputs numbers from a certain subset of the real numbers called the *domain* of the function; it prescribes to each input from its domain **exactly one** output.

From the mathematical point of view, the domain of the function f in Example 1.1.1, consists of all real numbers p. Indeed, according to the formula for the function $f(p)$, for any number p, the output $f(p)$ is defined and it is a number. From the practical point of view, a list price p cannot be negative. Therefore, for practical reasons, we restrict the domain of f to positive inputs p.

In an abstract setting, when variables do not have any real-life meaning, we commonly denote the independent variable by x, the dependent variable by y and a function by f:

$$y = f(x).$$

Example 1.1.2. Let $y = f(x)$ where $f(x) = 0.2x + 1$.

(a) What is the domain of f?

(b) Find f(2) and f(3).

(c) Find all inputs x for which the output y is equal to 0.

Solution. (a) For every input number x, we can multiply x by 0.2 and add 1. So the output $f(x) = 0.2x + 1$ is defined and it is a number for every x. Hence, the domain of f consists of all real numbers x.

(b) $f(2)$ is the value of the function or the output corresponding to the input $x = 2$. Thus, $f(2) = 0.2 \cdot 2 + 1 = 1.4$. The value of y corresponding to $x = 2$ is $y = 1.4$. Similarly, $f(3) = 0.2 \cdot 3 + 1 = 1.6$.

(c) We are looking for inputs x for which the output y is 0. In other words, we are looking for the value (or the values) of x which are solutions to the equation: $0 = 0.2x + 1$ or equivalently $0.2x + 1 = 0$. To find x, we use the standard techniques of solving equations: we add -1 to both sides and then we divide both sides of the equation by 0.2.

$$
\begin{aligned}
0.2x + 1 + (-1) &= 0 + (-1)\\
0.2x &= -1\\
x &= -\frac{1}{0.2}\\
x &= -5.
\end{aligned}
$$

Example 1.1.3. The amount of nicotine in a person's bloodstream, $N = f(t)$, in milligrams, is a function of time t, in hours, after the person finished smoking a cigarette.

(a) Identify the independent and the dependent variables of this function.

(b) In the statement $f(3) = 0.71$ what is the meaning of 3 and 0.71 in terms of time and nicotine? Give units of each of the two numbers.

(c) Explain the meaning of the statement $f(4) = 0.5$ in practical terms.

Solution. (a) N depends on t; N is a function of t, so N is the dependent variable, t is the independent variable.

(b) 3 is a value of t so its units are hours; 0.71 is a value of N so its units are mg. In practical terms, $f(3) = 0.71$ means that 3 hours after a cigarette is smoked there is 0.71 mg of nicotine left in the person's bloodstream.

(c) Similarly as in (b), $f(4) = 0.5$ means that 4 hours after a cigarette is smoked there is 0.5 mg of nicotine left in the person's bloodstream.

Example 1.1.4. In cases of strep throat, a daily pediatric dose of Amoxicillin, D, in milligrams depends on the weight of a child, w, in kilograms; that is, D is a function of w. Denote the function by g:

$$D = g(w).$$

As a general rule, the daily dose should be 50 milligrams for each kilogram of weight.

(a) Write a formula for the function $g(w)$.

(b) Calculate the daily dose for a child who weighs 8 kg.

(c) What does $g(11) = 550$ tell you in practical terms?

Solution. (a) $g(w)$ gives the dose for the weight of w kilograms. Since we give 50 mg for each kilogram, the dose $g(w)$ is 50 mg/kg $\cdot$ w kg. Hence, $g(w) = 50w$ milligrams.

(b) We calculate $g(8) = 50 \cdot 8 = 400$mg. The dose for a child who weighs 8 kilograms is 400 milligrams.

(c) To answer such questions, assign units to each number. 11 is a value for w so it is measured in kg. 550 is a value for $g(w)$ or D, so it is measured in mg. $g(w)$ gives the dose for a child who weighs w kg. Hence, the statement $g(11) = 550$ means in practical terms that a child weighing 8 kg should receive 550 mg of Amoxicillin in case of strep throat.

Example 1.1.5. A man leaves home and drives at 40 mph toward a hospital located 150 miles away from his home. t hours after he began driving his distance from the hospital d, in miles, is:

$$d = 150 - 40t.$$

(a) $d = f(t)$ is a function of t. $f(t) = ?$

(b) How far from the hospital is the man after 1.5 hours?

(c) When will the man arrive at the hospital?

Solution. (a) The formula for d shows how to obtain d from t. Hence, $f(t) = 150 - 40t$.

(b) The man's distance from the hospital at $t = 1.5$ is $f(1.5) = 150 - 40 \cdot 1.5 = 90$. After 1.5 hours, the man is 90 miles from the hospital.

(c) The man will arrive at the hospital when his distance d from the hospital is 0. We are looking for t such that $0 = 150 - 40t$ or equivalently $150 - 40t = 0$. We have to solve the equation for t:

$$\begin{aligned} 150 - 40t + 40t &= 0 + 40t \\ 150 &= 40t \\ \frac{150}{40} &= t \\ t &= 3.75. \end{aligned}$$

The man will arrive at the hospital 3 hours and 45 minutes after he left home.

Increasing and Decreasing Functions

In Example 1.1.3 we looked at the function $D = g(w)$ that gives the dose D of Amoxicillin for a patient weighing w kilograms. Notice that $g(w)$ increases as the weight w increases. In Example 1.1.5, the distance $d = f(t)$ from the hospital decreases as the number of hours t spent driving toward the hospital increases. The examples suggest the definition of increasing and decreasing functions.

Increasing and Decreasing Functions

- A function $f(x)$ is called *increasing* if the values $f(x)$ increase as x increases.
- A function $f(x)$ is called *decreasing* if the values $f(x)$ decrease as x increases.

So the dose of Amoxicillin is an increasing function of weight. The distance from the hospital is a decreasing function of time.

As we will see later, many functions are neither increasing nor decreasing through their domain. They may increase on some intervals and decrease on other intervals.

Functions as a Sequence of Operations

It is often useful to see a given function as a sequence of operations performed on an input in order to get the output. For example, the function, $f(x) = 0.2x + 1$ of Example 1.1.2 takes an input, multiplies the input by 0.2 and then adds 1 to the result. The function will perform those operations no matter what an input is. We can write it symbolically as:

$$f(\square) = 0.2 \cdot \square + 1.$$

Whatever input we feed into the function f — a number, an expression – whatever, the function will multiply the input by 0.2 and add 1.

Example 1.1.6. Let $g(x) = 3 - 4x$. Evaluate and simplify:

(a) $g(2 + h)$ (b) $g(1 - h) + g(1)$

Solution. The function $g(x)$ takes any input $\square$ — whatever it might be — multiplies the input by 4 and subtracts the result from 3: $g(\square) = 3 - 4 \cdot \square$. In (a), the input is an *algebraic expression*: $2 + h$. In any given context h may stand for a number or for a numerical variable. In any case, g takes $2 + h$ multiplies it by 4 and subtracts the result from 3. Thus, $g(2 + h) = 3 - 4(2 + h)$. To simplify, we expand $4(2 + h)$: $g(2 + h) = 3 - 8 - 4h = -5 - 4h$. We cannot simplify anymore. The final answer is $g(2 + h) = -5 - 4h$.

(b) We have:

$$g(1 - h) + g(1) = (3 - 4(1 - h)) + (3 - 4 \cdot 1) = (3 - 4 - 4(-h)) + (-1) = -1 + 4h - 1 = -2 + 4h.$$

Whatever h might be, $g(1 - h) + g(1) = -2 + 4h$.

Example 1.1.7. Find a formula for $f(x)$ if:

(a) f takes the square root of the input, multiplies the result by 3, and subtracts 1.

(b) f squares the input, adds 2, and takes the reciprocal of the result.

Solution. (a) Take an input x. We have to take the square root of the input which gives $\sqrt{x}$. Multiplication of the result by 3 gives $3\sqrt{x}$. Finally, we have to subtract 1 which gives $3\sqrt{x} - 1$. So the formula for f is $f(x) = 3\sqrt{x} - 1$. We would repeat the same sequence of operations for any other input $\square$.

(b) Take an input x. We square the input first which gives x^2. Adding 2 gives $x^2 + 2$. Now, we take the reciprocal of $x^2 + 2$ which is $\dfrac{1}{x^2 + 2}$. Hence, $f(x) = \dfrac{1}{x^2 + 2}$.

Example 1.1.8. What is the domain for the following functions:

(a) $f(x) = \dfrac{1}{x^2 - 1}$ (b) $g(x) = \sqrt{x - 3}$

Solution. (a) $f(x)$ is in the abstract context and has no practical meaning of any kind. Hence, the domain consists of all inputs x for which the output $f(x)$ is defined and it is a number. The output

$$f(x) = \frac{1}{x^2 - 1}$$

is defined unless the expression in the dominator is 0. That is, an input x is not in the domain if:

$$x^2 - 1 = 0$$

or equivalently

$$x^2 = 1.$$

$x^2 = 1$ for $x = 1$ and $x = -1$. Hence, the domain of $f(x)$ consists of all numbers x such that $x \neq 1$ and $x \neq -1$.

(b) Since we cannot have negative numbers under the radical, $g(x) = \sqrt{x - 3}$ is defined only for x such that $x - 3 \geq 0$ or equivalently $x \geq 3$. The domain of $g(x)$ is all real numbers x such that $x \geq 3$.

Practice Problems for Section 1.1

1. The value of a car, V, in dollars, is a function of the number of years t after purchase, $V = g(t)$.
   (a) What is the independent variable?
   (b) In what units is the independent variable measured?
   (c) What is the dependent variable?
   (d) In what units is the dependent variable measured?

2. The amount of caffeine, C, measured in mg, in a person's body t hours after drinking a cup of coffee is given by a function $C = f(t)$. What does each of the following statements tell you in practical terms; that is, in terms of time and caffeine? Answer in complete sentences and give units with each number.
   (a) $f(0) = 96$ (b) $f(5) = 48$ (c) $f(24) \approx 0$

3. The total cost of a meal in a restaurant, C, in dollars, as a function of the price of the meal, p, in dollars is given by:
   $$C = p + 0.20p$$
   where the term $0.20p$ corresponds to the 20% tip.
   (a) What is the input variable and what units it is measured in?
   (b) What is the output variable and what units it is measured in?
   (c) Calculate the total cost of a meal whose price is \$25.

4. A driver is heading to a faraway town. The amount of fuel, G, in gallons, left in the fuel tank is a function of the number of miles driven, m, during the trip; that is, $G = f(m)$.

   (a) What does the statement $f(70) = 6$ tell you in practical terms? What are units of the numbers 70 and 6?

   (b) What does the statement $f(200) = 1$ tell you in practical terms?

5. Let $f(x) = 2x + 1$. Evaluate:

   (a) $f(3)$ (b) $f(-4.5)$ (c) $f(0)$

6. Let $f(x) = 4 - 3x$. Evaluate and simplify if possible:

   (a) $f(3b)$ (b) $f(b+1)$ (c) $f(b+3) - f(3)$

7. Let $g(x) = 3x - x^2$. Evaluate:

   (a) $g(2)$ (b) $g(-2)$ (c) $g(\sqrt{2})$

8. Let $g(x) = x^2 + 3$. Evaluate and simplify if possible:

   (a) $g(h-1)$ (b) $g(h+1)$ (c) $g(b+1) - g(1)$

9. Let $f(x) = \dfrac{1}{x+1}$. Evaluate. If the value is undefined, say so.

   (a) $f(0)$ (b) $f(-1)$ (c) $f(-2)$

10. Let $f(x) = \dfrac{3}{x+2}$. Evaluate and simplify if possible.

    (a) $f(h-1)$ (b) $f(h+1)$ (c) $f(b) - 4f(1)$

11. The value of a car, V, in dollars, t years after purchase is given by the function $V = g(t)$, where $g(t) = 16500 - 1500t$.

    (a) What is the value of the car at the time of purchase?

    (b) What is the value of the car 5 years after purchase?

    (c) After how many years is the car worth nothing?

12. Find a formula for a function $h(x)$ for each of the following scenarios:

    (a) h multiplies the input by 7 then adds 5 to the result.

    (b) h adds 5 to the input and then multiplies the result by 7.

    Are the functions in (a) and in (b) the same?

13. Find a formula for a function $f(x)$ for each of the following scenarios:

    (a) f takes the square of the input, multiplies the result by 5, then subtracts 8.

    (b) f multiplies the input by 5, takes the square of the result, then subtracts 8.

    (c) f subtracts 8 from the input, takes the square of the result, then multiplies by 5.

    Are any of the three functions the same?

14. For each of the functions below, find its domain.

(a) $f(x) = x - 1$ (b) $f(x) = \dfrac{1}{x-1}$ (c) $f(x) = \dfrac{2}{x^2-4}$ (d) $f(x) = \sqrt{x}$ (e) $f(x) = x^2 - 4$

15. Since you see lightning immediately and it takes the sound of thunder about 5 seconds to travel a mile, you can calculate the distance between you and the lightning[1]. Count the number of seconds, S, between the flash of lightning and the sound of thunder. Then the distance, D, in miles, between you and the lighting is given by the function:

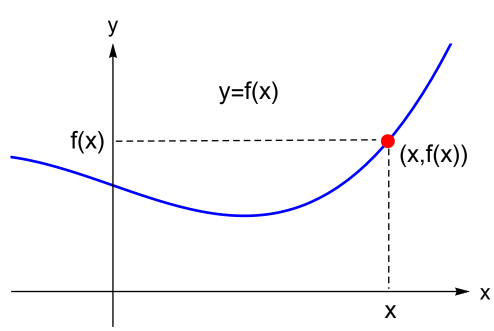
$$D = \frac{S}{5}$$

(a) Identify the independent and the dependent variable.

(b) How far is the lighting if you counted 4 seconds between the flash and the thunder?

1.2 The Graph of a Function

A great way to visualize a function is by drawing its graph. The graph of a function, say $y = f(x)$, is a curve on the xy-plane obtained as follows. Take an input x from the domain of the function f. Take the corresponding output y which is $f(x)$. Plot the point $(x, f(x))$ on the xy-plane. Ideally, we want to do it for all x in the domain. In practical terms, you cannot take all inputs x as most often there are infinitely many of them. If you are drawing the graph by hand, you will take relatively few points x, plot the corresponding points $(x, f(x))$ on the plane, and joint neighboring points by straight line segments. This should give you a rough approximation of the graph curve. Graphing devices like your graphing calculator do the exact same thing — they just take a great many points x. Typically, the graph forms a curve on the plane. It is the curve of points $(x, f(x))$ for which the second coordinate y is the value of the function at the first coordinate x. In other words, the graph is the curve of the points (x, y) that satisfy the equation $y = f(x)$:



It is useful to state a precise definition of the graph as similar definitions appear in other contexts in mathematics.

[1] https://www.weather.gov/safety/lightning-science-thunder, accessed: 6/26/20

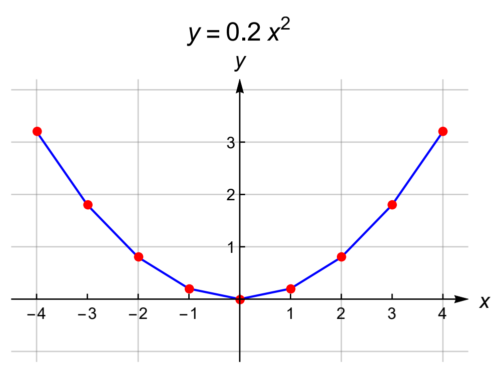
The Graph of a Function

The graph of a function $y = f(x)$ is the collection of all points (x, y) on the xy-plane for which $y = f(x)$. In other words, the graph is the collection of points $(x, f(x))$ for all inputs x in the domain of f.

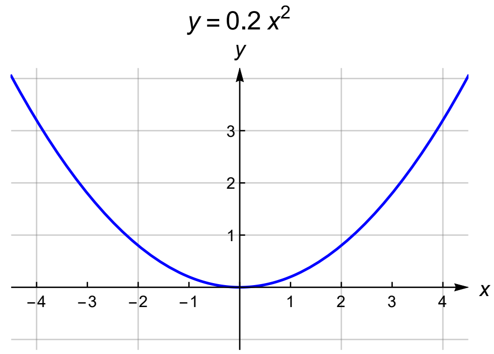
Example 1.2.1. Sketch the graph of the function $y = f(x)$, $f(x) = 0.2x^2$ by hand by plotting points on the graph for $x = 0, \pm 1, \pm 2, \pm 3, \pm 4$.

x	$y = f(x)$	$(x, f(x))$ pair
-4	$0.2 \cdot (-4)^2 = 3.2$	$(-4, 3.2)$
-3	$0.2 \cdot (-3)^2 = 1.8$	$(-3, 1.8)$
-2	$0.2 \cdot (-2)^2 = 0.8$	$(-2, 0.8)$
-1	$0.2 \cdot (-1)^2 = 0.2$	$(-1, 0.2)$
0	$0.2 \cdot (0)^2 = 0$	$(0, 0)$
1	$0.2 \cdot (1)^2 = 0.2$	$(1, 0.2)$
2	$0.2 \cdot (2)^2 = 0.8$	$(2, 0.8)$
3	$0.2 \cdot (3)^2 = 1.8$	$(3, 1.8)$
4	$0.2 \cdot (4)^2 = 3.2$	$(4, 3.2)$

We plot the points on the xy-plane and join the neighboring points by segments:



We obtained a pretty good approximation of the graph of $y = 0.2x^2$. Here is a better graph of $y = 0.2x^2$ plotted by a graphing software package:



In Example 1.2.1, the function $f(x)$ was given *algebraically*. That is, we had an algebraic formula for $f(x)$, $f(x) = 0.2x^2$. If a function is given algebraically, we can graph it as we did in Example 1.2.1.

Sometimes a function is given *graphically*. That is, all we know about a function is its graph. We can read a lot from the graph of a function.

Example 1.2.2. The amount of nicotine in a person's bloodstream, $N = f(t)$, in milligrams, is a function of time t, in hours, after the person finished smoking a cigarette. The graph of the function $f(t)$ is given in Figure 1.1:

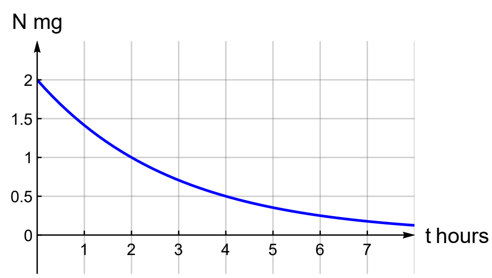


Figure 1.1

(a) How much nicotine is absorbed from a cigarette?

(b) How much nicotine is left in the bloodstream after 2 hours? After 4 hours?

Solution. (a) We see that $f(0) = 2$ (as the point $(0, 2)$ belongs to the graph of $f(t)$). Thus, 0 hours or right after the person finished smoking a cigarette, the amount of nicotine in the bloodstream is 2 mg. This is the amount absorbed from a cigarette.

(b) As time goes on, the amount of nicotine decreases. We see from the grid on the tN-plane that $f(2) = 1$; that is, 2 hours after a cigarette is smoked 1 mg is left in the bloodstream. (The point $(2, 1)$ lies on the graph.) $f(4) = 0.5$ – after 4 hours only 0.5 mg of nicotine is left.

Example 1.2.3. A woman is driving to visit with her family in a town 120 miles from her home. Let t be the time, in hours, after she left her home. Let d be the distance, in miles, to her destination. d is a function of t, $d = g(t)$. The graph of the function $g(t)$ is given in Figure 1.2:

Figure 1.2

(a) What is her distance from the destination 1 hour after she left home?

(b) Estimate the time when her distance from the destination is 60 miles.

(c) When does she reach her destination?

(d) How fast is she driving?

Solution. (a) We look at the graph. The point on the graph above $t = 1$ is $(1, 80)$. So the distance to the destination after 1 hour is 80 miles. In other words, $g(1) = 80$.

(b) We look for the point on the graph for which the second coordinate is $d = 60$. The t coordinate of that point seems to be at $t = 1.5$. After 1.5 hours, the woman is 60 miles from the destination. In other words, $g(1.5) = 60$; the point (1.5,60) lies on the graph.

(c) The woman reaches her destination when $d = 0$; that is, when $g(t) = 0$. Clearly from the graph, $g(3) = 0$. Hence the woman arrives at her destination after 3 hours.

(d) After 1 hour the distance dropped from 120 miles to 80 miles. After the next hour it dropped from 80 to 40 and from 40 to 0 during the hour after that. The woman travels at 40 mph.

Graphs of Increasing and Decreasing Functions

In graphical terms:

• A function is increasing if its graph **climbs** as the independent variable increases; that is, as we move from left to right.

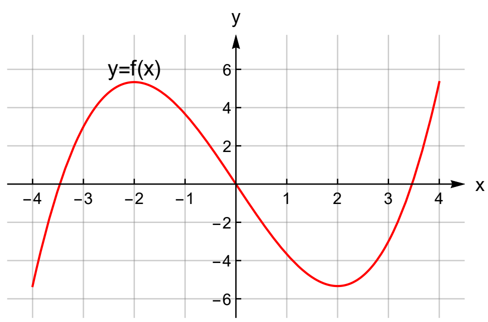
• A function is decreasing if its graph **falls** as the independent variable increases; that is, as we move from left to right.

The function $N = f(t)$ in Example 1.2.2 is decreasing — its graph is falling as t increases. The amount of nicotine decreases as time after smoking a cigarette increases. In Example 1.2.3 the distance to the destination $d = g(t)$ decreases as the time t spent driving increases. The function in Example 1.2.1 is neither increasing nor decreasing in its domain. At first, over the negative x-axis the graph is falling; the values $f(x)$ decrease as x increases. Then, over the positive x-axis, the graph is climbing; the values $f(x)$ increase as x increases. We can say that f is decreasing in the interval $x < 0$ and increasing in the interval $x > 0$.

Functions Increasing or Decreasing on Intervals

- A function $f(x)$ is increasing in an interval I if the values $f(x)$ increase as x increases along I.
- A function $f(x)$ is decreasing in an interval I if the values $f(x)$ decrease as x increases along I.

Example 1.2.4. Identify intervals on which the function $f(x)$ depicted below is increasing and intervals on which the function is decreasing.



Solution. The graph of $f(x)$ is climbing on the interval $-4 < x < -2$ and on the interval $2 < x < 4$. Hence, $f(x)$ is increasing in these intervals. The graph of $f(x)$ is falling and thus the function is decreasing in the interval $-2 < x < 2$.

Vertical Line Test

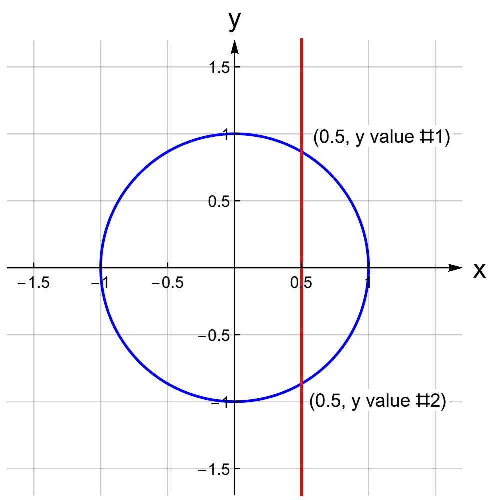
The graph of a function $y = f(x)$ is a curve on the xy-plane. Not every curve on the xy-plane is the graph of a function, though. The definition of a function requires that for each input x in the domain there is exactly one output y. There cannot be two outputs corresponding to one and the same input.

The Vertical Line Test is a simple visual way of determining if a given curve is or is not the graph of a function.

Vertical Line Test

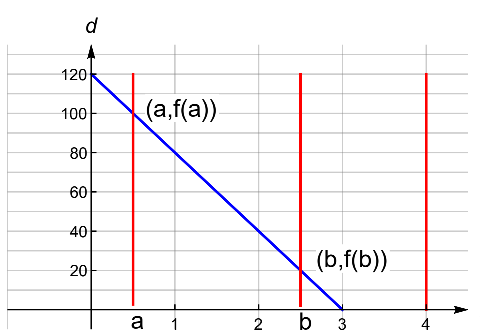
Let a curve on the xy-plane be given. If there is a vertical line that intersects the curve more than once, then the curve does not represent a function. If every vertical line intersects the curve at most once, then the curve represents the graph of a function.

Example 1.2.5. Consider a curve on the xy-plane which is the circle of radius 1 centered at the origin $(0, 0)$:



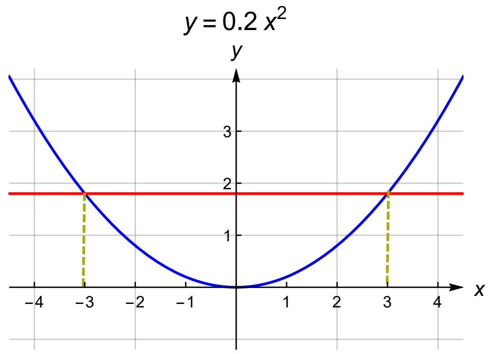
Take the vertical line corresponding to $x = 0.5$; that is, passing through the point $(0.5, 0)$ on the x-axis. This line intersects the circle at two points with two different values of y. Hence, the circle is not the graph of any function. Indeed, there would be two outputs corresponding to the input $x = 0.5$ which violates the definition of a function.

Example 1.2.6. Of course, it doesn't matter if the coordinates on the plane are labelled x, y or by any other letters. The idea is the same. Consider the following graph on the td-plane:



This is the same graph as the one we considered in Example 1.2.3. The segment is the graph of a function whose domain is the interval $0 \le t \le 3$. The Vertical Line Test confirms that. Indeed, every vertical line that passes through a value t on the horizontal axis such that $0 \le t \le 3$ crosses the segment at one point. A vertical line that passes through a value of t outside of the interval $0 \le t \le 3$ does not intersect the segment.

Note that, in general, a function can have the same output for two different inputs. In Example 1.2.1, the function $f(x) = 0.2x^2$ gives the same output $y = 1.8$ for $x = 3$ and $x = -3$. Still, for each x, we have only one value of y. Having the same output for two different inputs means that some **horizontal** lines intersect the graph more than once which is perfectly fine for a function:



We will look at other properties of graphs of functions in Section 1.4.

Practice Problems for Section 1.2

1. Create a table of values and sketch the graph of the function $y = x^3$ for $-2 \le x \le 2$. Then use your calculator or any other graphing utility to check your graph.

2. The total cost of a meal in a restaurant, C, in dollars, as a function of the price of the meal, p, in dollars is given by:

$$C = p + 0.20p$$

   where the term $0.20p$ corresponds to the 20% tip. Create a table of values and sketch the graph of the function $C = p + 0.20p$ for $10 \le p \le 20$. Then use your calculator or any other graphing utility to check your graph.

3. Use the graph of the function $y = f(x)$ below, to estimate:

   (a) $f(2)$ (b) $f(3)$ (c) $f(-1)$

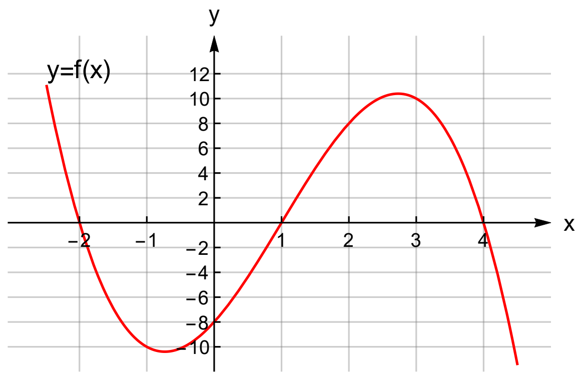


Figure 1.3

4. For the function $y = f(x)$ in Figure 1.3, estimate all points x for which $f(x) = 0$.

5. For the function $y = f(x)$ in Figure 1.3, estimate all points x for which $f(x) = 8$.

6. A driver of a 2019 Toyota Corolla fills up his gas tank and embarks on a highway trip. The amount of gas left in the tank, G, in gallons, is a function of the number of miles m driven, $G = g(m)$. The graph of $g(m)$ is given below.

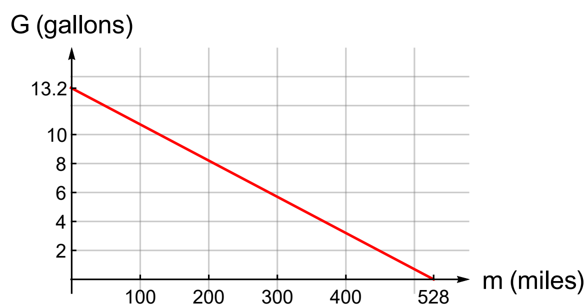


Figure 1.4

Use the graph to answer the following questions.

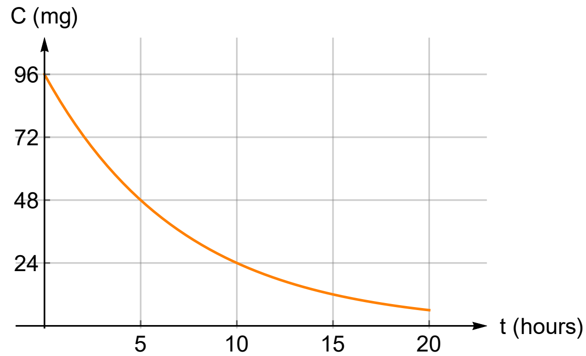
(a) What is the fuel tank capacity of the 2019 Toyota Corolla?

(b) How much fuel is left after 200 miles?

(c) What happens after 528 miles?

(d) What is the fuel efficiency of the 2019 Toyota Corolla on the highway?

7. The amount of caffeine remaining in the body, C, in milligrams, t hours after drinking a cup of coffee, $C = C(t)$ is given by the graph below:

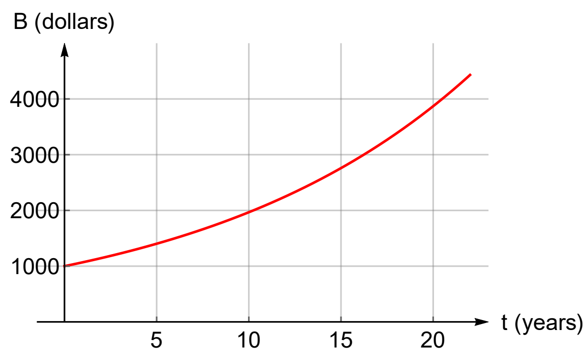


(a) How much caffeine was absorbed into the bloodstream from the cup of coffee?

(b) How much caffeine is left after 5 hours? After 10 hours?

(c) Is the function $C(t)$ increasing, decreasing or neither in the interval $t \geq 0$?

8. A man deposits money into a savings account. His balance $B(t)$, in dollars, after t years is given by the graph below:

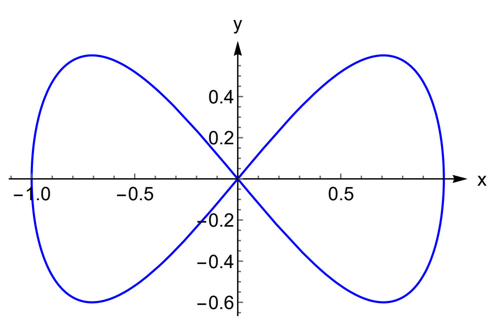


(a) What was his initial deposit?

(b) How much money was in his account after 10 years? After 20 years?

(c) Is the function $B(t)$ increasing, decreasing or neither in the interval $t \geq 0$?

9. Is the curve below the graph of a function $y = f(x)$? Explain!



10. The graph of a function $y = f(x)$ is given in Figure 1.5:

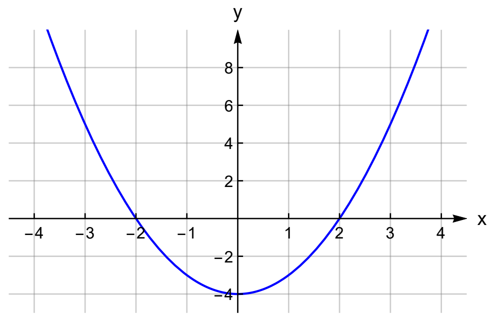


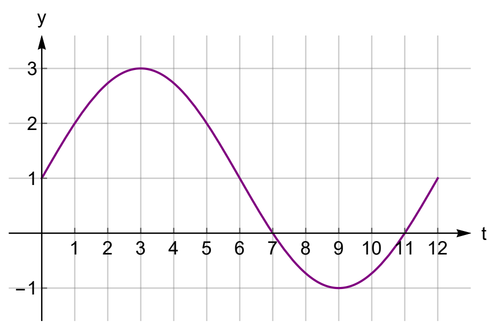
Figure 1.5

(a) Estimate $f(0)$.

(b) Estimate all values of x for which $f(x) = 0$.

(c) Estimate all values of x for which $f(x) = 5$.

11. For the function $y = f(t)$ whose graph is depicted below, identify intervals on the t-axis on which the function is increasing and the intervals on the t-axis on which the function is decreasing.



1.3 Functions Given Numerically

In the previous section we saw that if we have a function given algebraically — by a formula — we can easily create any table of values for that function. We can use a table of values to sketch the graph of the function which provides a visual representation of the function. We can use the graph to see the behavior of the function that may not be readily visible from the formula alone.

When we study a real-life process, which inevitably involves a function of some kind, we are rarely provided in advance a formula for a function that represents — or *models* — our process. Typically, we take measurements, collect and tabulate numerical data for the process we study. After having done that, we try to find a formula for a function that represents our numerical data. In real life the road leads from a table of values to a formula for a function and not the other way around. Finding a mathematical description of a real-life process is called *mathematical modeling.*

A function that is given by its table of values is said to be given *numerically.*

Example 1.3.1. If you ever kept overripe bananas in your house a bit too long, you were probably invaded by fruit flies. And you probably noticed how fast the population of fruit flies in your house was increasing. It is hard to count the number of fruit flies in a house but it can be done in a laboratory.

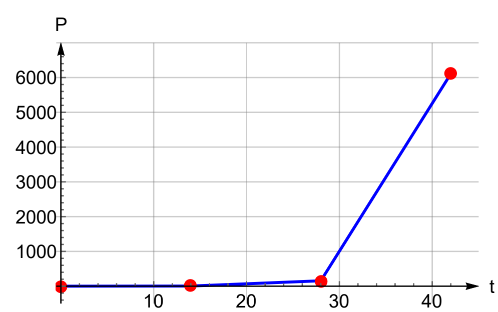
Suppose that a population of fruit flies in a laboratory experiment is initially 100 flies and it begins to grow. The size of the population, P, measured in **thousands** of flies, is a function of time t since the experiment began; that is, $P = f(t)$. Given how fast fruit flies multiply, it is reasonable to measure t in days. We are not given a formula for $f(t)$ describing the growth of our population and, in general, that growth and therefore the function $f(t)$ depends on many factors created in our lab, including the temperature, food provided etc.. We observe the population and take measurements every 14 days:

t (in days)	0	14	28	42	56
$P = f(t)$ (in thousands)	0.1	3.94	155.12	6106.99	240432.20

Table 1.1

(Measuring the population every 14 days makes sense given the fruit fly reproductive cycle.) We see that $f(t)$ is an increasing function and it increases very fast. After 56 days, the initial population of 100 flies grew to 240432.2 thousand flies; that is, 240,432,200 flies. Can we find a formula for a function $f(t)$ so that Table 1.1 is a table of values for $f(t)$? In other words, can we find a mathematical model for the population growth? Finding a formula for a function that corresponds to experimental data, even if only approximately, may be difficult, very often impossible. In this example, a function $f(t)$ that corresponds to Table 1.1 can be found as the growth of the population is exponential. We will learn about exponential functions in Chapter 5. For now, we have only a numerical representation of $f(t)$.

Given a numerical representation of a function, we can sketch its approximate graph. We graph the function $f(t)$ for t between 0 and 42. The value at $t = 56$ is so large that it squeezes the graph vertically too much and makes it hard to read.



Example 1.3.2. On December 31, 2019 the price of Exxon Mobile stock (NYSE: XOM) was changing during the day. Here is the price of the stock every hour from 10 am to 4 pm:

Time	10 am	11 am	12 pm	1 pm	2 pm	3 pm	4 pm
Price in dollars	69.35	69.17	69.41	69.38	69.32	69.32	69.78

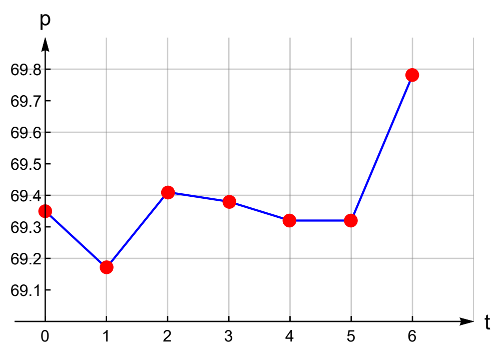
Table 1.2

The price of the stock is a function of time. To express this function in a more mathematically friendly form, denote by t the time, in hours, since 10 am so 10 am is $t = 0$, 11 am is $t = 1$, 4 pm is $t = 6$, and so on. Denote by E the price of the stock, in dollars. Denote the price function by p. Hence, $E = p(t)$ is the function that gives the price of Exxon Mobile stock at time t on that particular day. $E = p(t)$ is given numerically by:

t	0	1	2	3	4	5	6
$p(t)$	69.35	69.17	69.41	69.38	69.32	69.32	69.78

Table 1.3

We can use the data to sketch the graph of $E = p(t)$:



The function $p(t)$ is neither increasing nor decreasing. The graph goes up and down as t increases. Finding a formula for the function $p(t)$ that models fluctuations of the price of a stock doesn't seem possible and we will not try to do it.

Example 1.3.3. A woman comes to a gym to exercise. After t minutes on a treadmill, her pulse (heart rate), H, in beats per minute, is:

t (minutes)	0	2	4	6	8	10
H (bpm)	80	84	88	92	96	100

Table 1.4

The woman's pulse is a function of time: $H = H(t)$. The function is given numerically; pulse readings are taken every 2 minutes. For the sake of simplicity, we often denote the dependent variable and the function by the same letter. We write: $H = H(t)$ which means H is a function of t. The dependent variable and the function are denoted here by the same letter H.

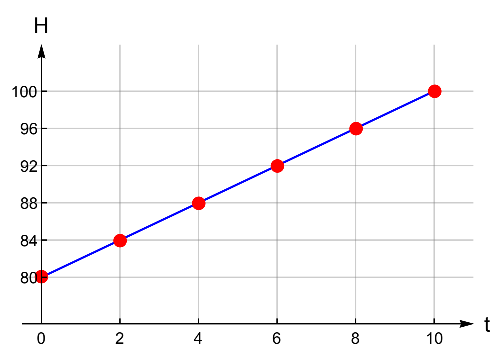
Can we find a formula for $H(t)$? At this point, we can only guess and use common sense. We haven't developed any techniques yet. Look at the data in Table 1.4. The pulse H increases as time t increases. That is to be expected. How does H increase? The initial value of H, $H(0)$ is 80. During the first two minutes from $t = 0$ to $t = 2$, the pulse increased by 4 bpm from 80 to 84; during the next two minutes from $t = 2$ to $t = 4$, the pulse increased again by 4 bpm from 84 to 88. We look at each of the two-minute intervals and see that the pulse increases by 4 bpm every two minutes: 88 to 92, 92 to 96, and 96 to 100. Can you guess a possible formula for $H(t)$? 4 beats every two minutes so perhaps 2 beats every 1 minute. We start from 80 and then add 2 for every one-minute change in t. The formula that reflects that is:

$$H(t) = 80 + 2t.$$

Let's check if the values of $H(t)$ reflect the data in Table 1.4:

$H(0) = 80 + 2 \cdot 0 = 80$, $H(2) = 80 + 2 \cdot 2 = 84$, $H(4) = 80 + 2 \cdot 4 = 88$, $H(6) = 80 + 2 \cdot 6 = 92$, $H(8) = 80 + 2 \cdot 8 = 96$, and $H(10) = 80 + 2 \cdot 10 = 100$.

Yes! The function $H(t)$ has all the right values as listed in Table 1.4. Note that the formula $H(t) = 80 + 2t$ gives a *possible* formula for $H(t)$. The table does not tell us anything about the values of $H(t)$ between measurements. Numerical representation of functions leaves a lot of information out. Here is an approximate graph of the function $H = H(t)$ based on the numerical data:



Example 1.3.4. Function $y = f(x)$ and $y = g(x)$ are given numerically below. Find possible formulas for $f(x)$ and $g(x)$.

(a)

x	0	1	2	3	4	5
$f(x)$	0	-1	-4	-9	-16	-25

Table 1.5

(b)

x	0	1	2	3	4	5
$g(x)$	80	78	76	74	72	70

Table 1.6

Solution. (a) We notice that the values of $f(x)$ for each x given in the table are negative and have the magnitude equal to x^2. Hence, a possible function represented by Table 1.5 is $f(x) = -x^2$. We can easily check that $f(x) = -x^2$ works by substituting $x = 0, 1, 2, 3, 4, 5$ into $f(x)$ and calculating the corresponding values. Those are indeed: $0, -1, -4, -9, -16, -25$.

(b) At $x = 0$, $g(0) = 80$. Then, as x increases, $g(x)$ decreases in a very regular fashion: for each increase of 1 in x, $g(x)$ decreases by 2. A formula that gives exactly such behavior is:

$$g(x) = 80 - 2x.$$

We can check: $g(0) = 80$, $g(1) = 80 - 2 \cdot 1 = 78$, $g(2) = 80 - 2 \cdot 2 = 76$, $g(3) = 80 - 2 \cdot 3 = 74$, $g(4) = 80 - 2 \cdot 4 = 72$, and, finally, $g(5) = 80 - 2 \cdot 5 = 70$.

For now, we can only try to guess formulas for numerically given functions. As we go along and study various families of functions, we will develop systematic methods of finding formulas for functions given numerically, at least some types of them.

Practice Problems for Section 1.3

1. A function $f(x)$ is given numerically below:

x	0	2	4	6	8	10
$f(x)$	1.5	3.7	1.5	4.2	1.5	0

Table 1.7

Find the outputs: (a) $f(4)$ (b) $f(2)$ (c) $f(8)$.

2. Let $f(x)$ be the function in Table 1.7.

(a) Find all inputs x for which $f(x) = 1.5$.

(b) Find all inputs x for which $f(x) = 3.7$.

3. Guess a formula for the function $g(x)$ given numerically in the table below.

x	-2	-1	0	1	2	3	4
$g(x)$	-8	-1	0	1	8	27	64

4. Guess a formula for the function $h(t)$ given numerically in the table below. Fill in the missing values.

t	-2	-1	0	1	2	3	4	5	6
$h(t)$	4	2	0	-2	-4	-6	-8	?	?

5. Here is the data for the world population[2] between 2010 and 2018, in billions:

Year	2010	2011	2012	2013	2014	2015	2016	2017	2018
Population	6.957	7.041	7.126	7.211	7.296	7.380	7.464	7.548	7.631

Table 1.8

To study the population from the mathematical point of view, it is convenient to rewrite the data in terms of the independent variable t which stands for the number of years since 2010 and write the population, P, in billions, as a function $P = P(t)$ of years since 2010. Based on Table 1.8 fill in the missing values:

t	0	1	?	3	4	?	6	7	8
$P(t)$	6.957	7.041	?	7.211	7.296	7.380	7.464	?	7.631

6. A driver of an SUV fills up his gas tank and starts a highway trip to a faraway city. Let G, in gallons, be the amount of gas left in the tank after driving d miles. Of course G is a function of d; that is, $G = G(d)$. Here is partial data about the function:

d	0	60	120	180	240	300	360	420	480
$G(d)$	17	15	13	11	9	7	?	?	?

Find gas mileage of the SUV; that is, the number of miles the SUV gets per gallon. Fill in the missing numbers.

7. The population of a village, $P = P(t)$, in the number of people, t years after the year 2000 was:

t	0	1	2	3	4	5
$P(t)$	110	160	230	310	410	530

Table 1.9

Is the population $P(t)$ increasing or deceasing? Is it increasing faster and faster or slower and slower? Explain how you arrive to your conclusions based on the data in Table 1.9.

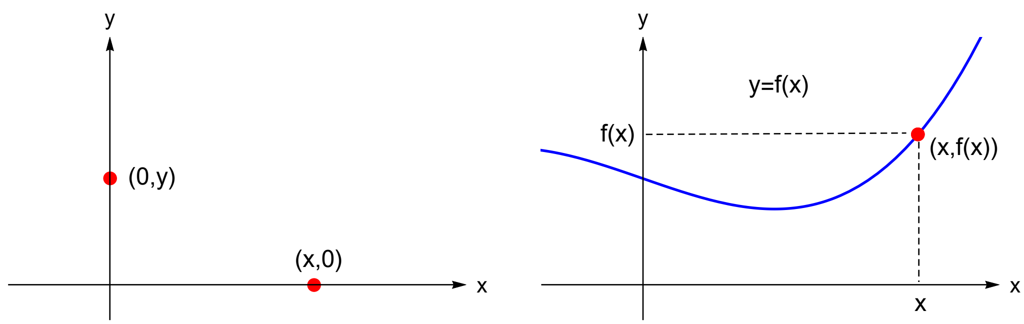
8. Use Table 1.9 to sketch the graph of the population function $P(t)$ (by hand or using a graphing utility). Use the graph to answer the question: Is the population increasing faster and faster or slower and slower? Explain your answer in terms of the graph and points on the graph.

[2]https://www.worldometers.info/world-population/world-population-by-year/, accessed: 6/28/20

1.4 Vertical and Horizontal Intercepts

The points where the graph of a function crosses the horizontal axis or the vertical axis are of special interest. They are called the horizontal and the vertical intercepts. If the independent variable is named x and the dependent variable is named y, the horizontal and vertical intercepts are also called x-intercepts and y-intercepts, respectively. We will examine intercepts in this section.

Notice that every point on the xy-plane that lies on the y-axis has the x-coordinate 0. Every point on the xy-plane that lies on the x-axis has the y-coordinate 0. Recall also that every point (x, y) that lies on the graph of a function $y = f(x)$ is of the form $(x, f(x))$ as the y coordinate has to be equal to the value of the function $f(x)$.

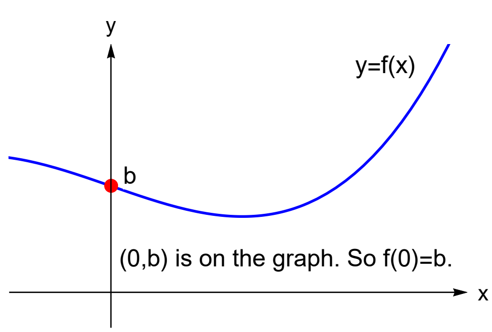


Let a function $y = f(x)$ be given.

The Vertical Intercept

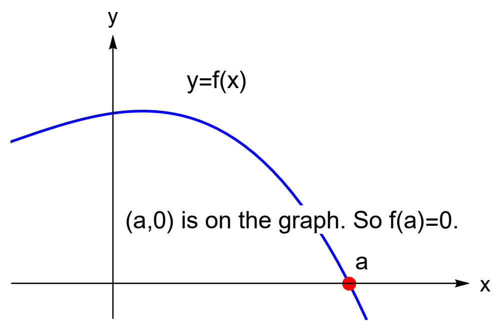
Suppose that the graph of the function $y = f(x)$ crosses the y-axis at $y = b$; that is, at the point $(0, b)$. Then b is called the y-intercept of the function f. Since $(0, b)$ is on the graph of the function, $b = f(0)$. So the y-intercept is simply the value of the function at $x = 0$. If $x = 0$ is in the domain of f, we have exactly one y-intercept. If $x = 0$ is not in the domain, the graph of f does not cross the y-axis and there is no y-intercept.

To find the vertical intercept of $f(x)$, we simply evaluate $f(0)$.



Horizontal Intercepts

Suppose that the graph of $f(x)$ crosses the x-axis at $x = a$; that is, at the point $(a, 0)$. Then a is called an x-intercept of the function f. Since the point $(a, 0)$ is on the graph $y = f(x)$, 0 is the value of the function f at a, so $0 = f(a)$ or equivalently $f(a) = 0$. Hence, x-intercepts are values of x at which $f(x) = 0$. A function may have many x-intercepts. They are often called the *zeros* of the function f.



To find horizontal intercepts of $f(x)$, we have to solve for x the equation $f(x) = 0$.

Intercepts from a Formula

Example 1.4.1. Consider the function $y = f(x)$ where $f(x) = x^2 - 4$. Find:

(a) The vertical intercept (b) Horizontal intercepts

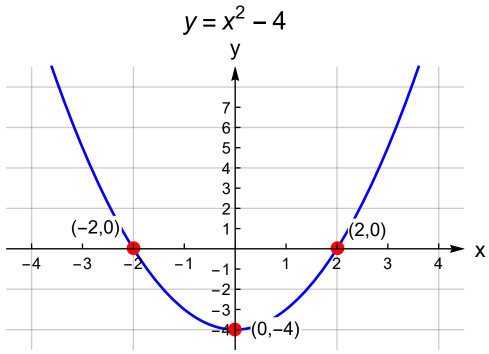
Solution. (a) To find the y-intercept, we evaluate $f(0)$: $f(0) = 0^2 - 4 = -4$. The y-intercept is -4. The graph of $f(x) = x^2 - 4$ crosses at the y-axis at $y = -4$; that is, at the point $(0, -4)$.

(b) To find x-intercepts, we have to solve the equation $f(x) = 0$ which for our function is $x^2 - 4 = 0$. We use the standard techniques of solving equations.

$$\begin{aligned} x^2 - 4 &= 0 \\ x^2 &= 4 \\ x &= \pm\sqrt{4} \\ x &= \pm 2 \end{aligned}$$

So we have two horizontal intercepts, $x = -2$ and $x = 2$. These are two zeros of the function: $f(-2) = 0$, $f(2) = 0$.

The graph of the function clearly shows the intercepts and the corresponding points on the xy-plane:



Example 1.4.2. A tank that holds 60 gallons of water springs a leak. Water is leaking out at the rate of 3 gallons per hour. Let $A = w(t)$ be the amount of water in the tank, in gallons, t hours after the leak started. The function $w(t)$ that reflects the initial amount of 60 gallons and the decrease of 3 gallons per hour is

$$A = w(t) = 60 - 3t.$$

Find vertical and horizontal intercepts of the function $A = w(t)$ and interpret them in practical terms.

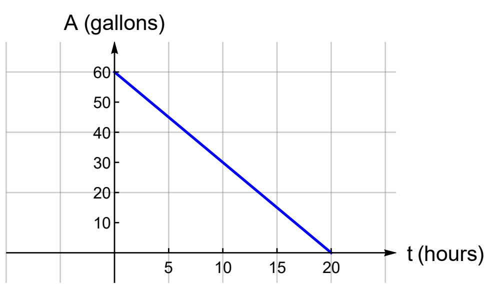
Solution. To find the vertical intercept, the A-intercept, of the function $A = w(t)$, we evaluate $w(0) = 60$. The vertical intercept is $A = 60$. Since 60 is the value of w at $t = 0$, it is measured in gallons and it gives the initial amount of water in the tank.

To find horizontal intercepts, we have to solve for t the equation: $w(t) = 0$; that is:

$$\begin{aligned} 60 - 3t &= 0 \\ 60 &= 3t \\ 20 &= t \\ t &= 20 \end{aligned}$$

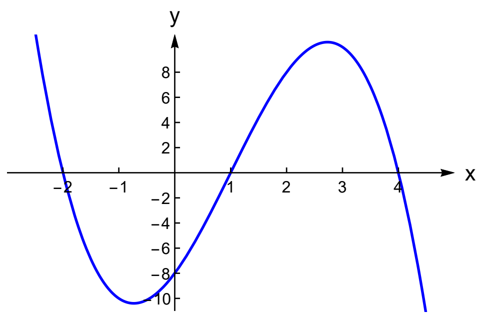
The function has one horizontal intercept at $t = 20$. At $t = 20$, 20 hours after the leak started, the amount of water $A(20)$ left in the tank is $A = w(20) = 0$. In practical terms, it will take 20 hours until the tank becomes empty.

Here is the graph of the function $A = w(t)$. The intercepts are clearly visible on the graph and their meaning is clear as well.



Intercepts from a Graph

Example 1.4.3. Use the graph of a function $y = f(x)$ below to estimate vertical and horizontal intercepts.



Solution. The graph shows that $f(0) = -8$. Hence, the vertical intercept is $y = -8$. The point on the xy-plane where the graph crosses the y-axis is $(0, -8)$.

The graph crosses the x-axis at $x = -2, 1, 4$. Those are the x-intercepts of $f(x)$. These are zeros of the function $f(x)$; that is, $f(x) = 0$ for $x = -2, 1, 4$. The corresponding points of intersection on the xy-plane are $(-2, 0)$, $(1, 0)$, $(4, 0)$.

Intercepts of a Function Given Numerically

Example 1.4.4. Every summer the depth of water in a reservoir is measured weekly over the 10 week period beginning with July 1. Let D be the depth of water, in meters, t time, in weeks, since July 1. Of course, D is a function of t, $D = D(t)$. Here are the readings from 2019.

t (weeks)	0	1	2	3	4	5	6	7	8	9	10
$D(t)$ (meters)	10	8	7	4	2	0	3	4	3	0	4

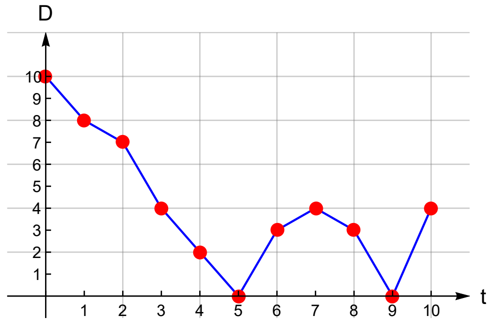
Table 1.10

What are the vertical and horizontal intercepts given by the table? What is their practical meaning?

Solution. The vertical intercept is $D(0)$. We have that value in the table: $D(0) = 10$. Hence, the vertical intercept, the D-intercept, is 10. In practical terms, the intercept tells us that water depth on July 1, 2019 was 10 meters.

The horizontal intercepts are values of t for which $D(t) = 0$. The depth is 0 at $t = 5$ and $t = 9$. In practical terms, 5 weeks after July 1 and 9 weeks after July 1, the reservoir is empty.

Below is the graph of the data in Table 1.10. We can clearly see the intercepts.



Practice Problems for Section 1.4

In Problems 1-7 find horizontal and vertical intercepts or state that they don't exist.

1. $y = 3x - 1.5$
2. $f(x) = 1 - x^2$
3. $g(t) = 5.4 - 0.2t$
4. $y = 2x^3 + 16$
5. $h(m) = 2\sqrt{m} - 3$
6. $y = \dfrac{1}{x}$
7. $y = \dfrac{1}{x-1}$

In Problems 8-10 find horizontal and vertical intercepts visible in the given tables of values.

8.

x	-2	-1	0	1	2	3
y	3	0	2	2.5	0	4

9.

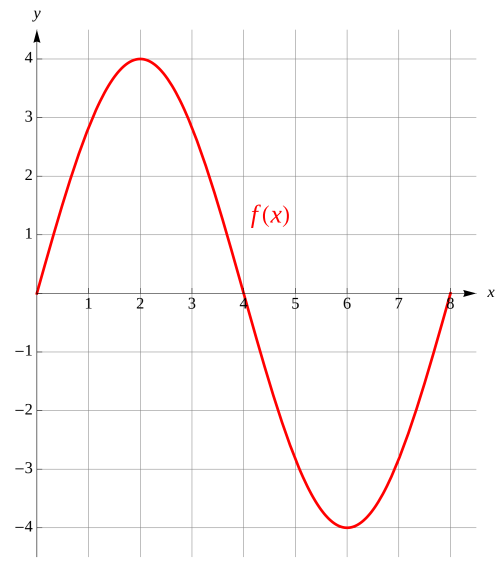
t	0	2	4	6	8	10
$f(t)$	5	6	8	4	8	4

10.

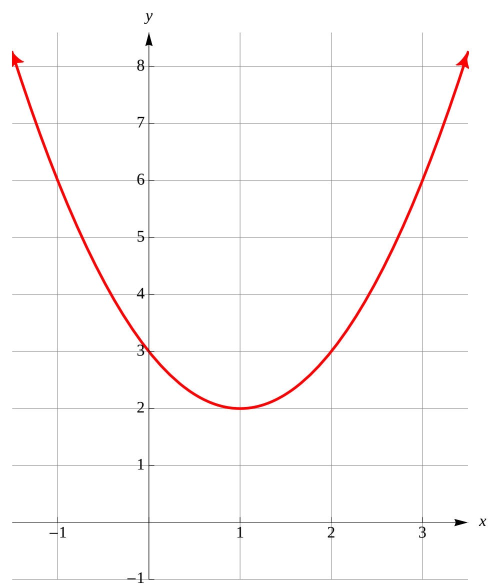
x	3	4	5	6	7	8
$g(x)$	-5	-4	-5	-2	-2	0

In Problems 11-12 estimate horizontal and vertical intercepts given the graph of a function.

11.



12.



13. A ball is dropped from a cliff above a lake. The ball's height above the surface of the lake, $h(t)$, in feet, t seconds after the ball is dropped, is given by:

$$h(t) = 300 - 16t^2$$

Find the vertical intercept of the function $h(t)$ and the horizontal intercepts for which $t \geq 0$. (Round off to two decimal places.) Interpret the intercepts in practical terms.

14. The value of a car $V(t)$, in dollars, t years after the car was purchased is:

$$V(t) = 21500 - 1500t$$

Find vertical and horizontal intercepts of the function $V(t)$ and interpret them in terms of dollars and years.

1.5 The Average Rate of Change

The notion of a rate of change is central to mathematics and its applications. Roughly speaking, a rate of change tells us how fast and in what manner a given quantity is changing with respect to another changing quantity.

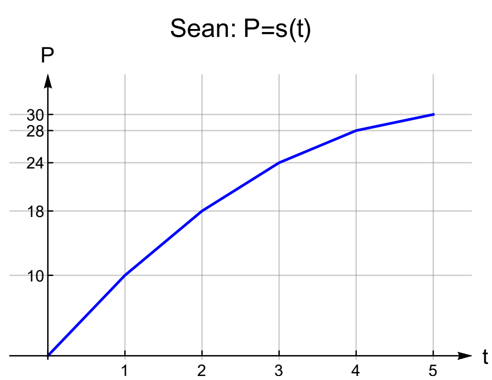
Suppose that a population of a town grows by 200 people each year. In other words, the rate of change of the population with respect to time is constant and equal to 200 people/year.

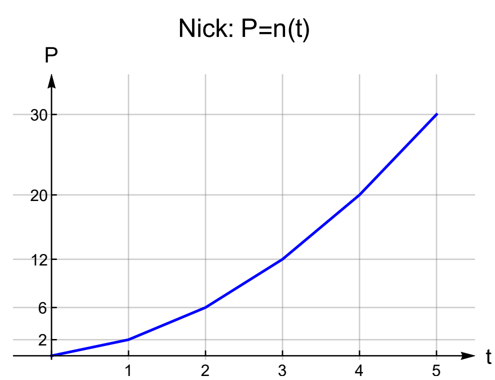
Recall Example 1.1.4: A daily pediatric dose of Amoxicillin, D, in milligrams, depends on the weight of a patient, w, in kilograms. More precisely, $D = 50w$. How fast does the dose increase as the weight increases? The formula for D clearly shows that for 1 kilogram increase in weight, the dose increases by 50 milligrams. So the rate of change of D with respect to weight w is 50 milligrams/kilogram.

Suppose that the value of your car changes at the rate -1500 dollars/year. The rate of change is negative. Each year the value of the car changes by -1500 dollars; that is, the value of the car decreases by 1500 each year. A quantity that changes at a negative rate is decreasing.

It is all straightforward for as long as the rate of change of the dependent variable with respect to the independent variable is constant as in the examples above. In real-life processes it is rarely the case. Let's look at the following example.

Example 1.5.1. Two students, Sean and Nick, come to the library to study for an upcoming precalculus exam. After t hours of uninterrupted study, Sean has mastered $P = s(t)$ pages from the textbook that were new to him when he came to the library; Nick has mastered $P = n(t)$ new pages. The graphs of the two functions $s(t)$ and $n(t)$ are as follows:





According to the graphs, both students stayed in the library for 5 hours. During the 5-hour session both students learned 30 new pages as $s(5) = 30$ and $n(5) = 30$. Are their learning patterns identical? How fast was each student learning during the 5-hour session?

Their learning patterns are clearly not identical. Look at the first hour. During the first hour, Sean has mastered $s(1) - s(0) = 10 - 0 = 10$ new pages. A lot! During the first hour, Sean was learning fast. During the first hour, Nick has mastered only $n(1) - n(0) = 2 - 0 = 2$ new pages. During the first hour Nick was learning slowly. Look at the last hour. During the last hour, Sean has learned only $s(5) - s(4) = 30 - 28 = 2$ pages. During the last hour, Nick has learned $n(5) - n(4) = 30 - 20 = 10$ new pages. So we see that Sean was learning fast at the beginning but slower and slower as time went on. Nick the other way around. Nick started very slowly and picked up the pace as time went on.

Does it make sense to ask how fast — or at what rate in pages/hour – each student was learning during the 5-hour session? It doesn't make sense. How fast was each student learning? When exactly: At the beginning? At the end? For both students the rate at which they were learning during the session was changing. Sean was learning slower and slower. Nick was learning faster and faster. The rate, in pages per hour, at which each of them was learning was changing during the 5 hours.

What does make sense to ask is how fast each student was learning during the 5-hour session **on average**. Sean was learning fast at first, slower later, but overall he mastered 30 pages in 5 hours. So he was learning at the **average** rate of $\frac{30}{5} = 6$ pages/hour. Similarly for Nick. He was learning slowly at first, then faster, but he also learned 30 pages in 5 hours. So he, too, was learning at the **average** rate of $\frac{30}{5} = 6$ pages/hour.

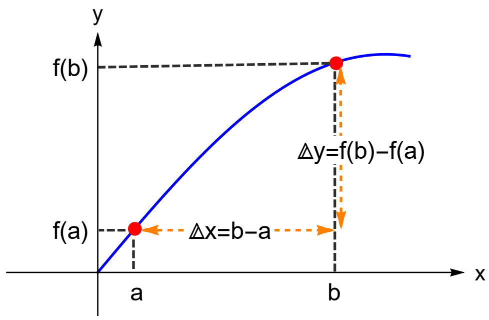
Later, in Calculus, you will learn about more granular concepts of rate of change that capture moment to moment changes. For now, the **average** rate of change is the best we can do.

The Average Rate of Change

Let y be a function of x, $y = f(x)$. Let $x = a$ and $x = b$ be given. The average rate of change of y between $x = a$ and $x = b$ is:

$$\text{Average rate of change} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Customarily, the symbol Δ (the Greek letter delta) is used to mean "change in". So, Δy stands for a change in y, Δx for a change in x. In the definition above, Δx stands for a change in x between $x = a$ and $x = b$ which is $\Delta x = b - a$; Δy stands for the change in $y = f(x)$ that takes place when x changes between $x = a$ and $x = b$. That change is $\Delta y = f(b) - f(a)$. The graph below illustrates the changes in x and y.



Example 1.5.2. Let $y = f(x) = x^2 - 2$. Find the average rate of change of y between $x = 1$ and $x = 3$.

Solution. We use the definition of the average rate of change with $a = 1$, $b = 3$ and $y = f(x) = x^2 - 2$:

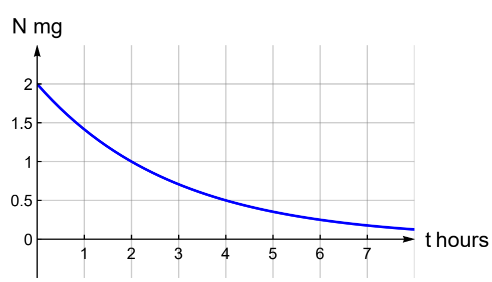
$$\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(3) - f(1)}{3 - 1} = \frac{(3^2 - 2) - (1^2 - 2)}{3 - 1} = \frac{7 - (-1)}{2} = \frac{8}{2} = 4.$$

The average rate of change of y between $x = 1$ and $x = 3$ is 4.

This is not an applied example so x and y don't have real-life units. In general, the average rate of change $\frac{\Delta y}{\Delta x}$ is measured in $\frac{\text{units of y}}{\text{unit of x}}$. In this example, we can simply write that the average rate of change of y between $x = 1$ and $x = 3$ is $4 \, \frac{\text{units of y}}{\text{unit of x}}$.

Let's revisit Example 1.2.2 about nicotine leaving the body after a cigarette is smoked.

Example 1.5.3. The amount of nicotine in a person's bloodstream, $N = f(t)$, in milligrams, is a function of time t, in hours, after the person finished smoking a cigarette. The graph of the function $f(t)$ is:



Use the graph to find the average rate of change of N between:

(a) $t = 0$ and $t = 4$; (b) $t = 2$ and $t = 4$.

Give units with your answers.

Solution. (a) To find the average rate of change between $t = 0$ and $t = 4$, we need values of the function $f(t)$ at $t = 0$ and at $t = 4$. We find these values from the graph. At $t = 0$, the corresponding value of $N = f(t)$ is 2 milligrams. At $t = 4$, the value of $f(t)$ is 0.5 milligrams. Thus:

$$\text{Average rate of change} = \frac{\Delta N}{\Delta t} = \frac{f(4) - f(0)}{4 - 0} = \frac{0.5 - 2}{4 - 0} = \frac{-1.5}{4} = -0.375.$$

Here the variables do have units: N is in milligrams, t is in hours. So ΔN is in milligrams, Δt is in hours and the average rate of change is in $\frac{\text{milligrams}}{\text{hour}}$ written as $\frac{\text{mg}}{\text{h}}$.

Therefore, the average rate of change of N between $t = 0$ and $t = 4$ is $-0.375\,\frac{\text{mg}}{\text{h}}$. The rate is negative as the amount of nicotine decreased between $t = 0$ and $t = 4$. In the interval $0 \le t \le 4$, nicotine was leaving the body at the average rate of $-0.375\,\frac{\text{mg}}{\text{h}}$.

(b) We need values $f(2)$ and $f(4)$. We read them from the graph: $f(2) = 1$ and $f(4) = 0.5$. Hence:

$$\text{Average rate of change} = \frac{\Delta N}{\Delta t} = \frac{f(4) - f(2)}{4 - 2} = \frac{0.5 - 1}{2} = \frac{-0.5}{2} = -0.25\,\frac{\text{mg}}{\text{h}}.$$

In the interval $2 \le t \le 4$, nicotine was leaving the body at the average rate of $-0.25\,\frac{\text{mg}}{\text{h}}$.

Example 1.5.4. A population P, of a small village, in number of people, at time t, measured in years since the year 2010, is given by $P = P(t)$. Here are the annual readings:

t (years)	0	1	2	3	4	5	6	7	8	9	10
$P(t)$ (people)	150	160	140	180	210	200	250	290	300	350	400

Table 1.11

(a) Find the change in the population ΔP between $t = 2$ and $t = 6$. What are the units of ΔP?

(b) Find the average rate of change in the population between $t = 2$ and $t = 6$. What are the units of this rate of change?

Solution. (a) The change in P between $t = 2$ and $t = 6$, ΔP, tells us by how much the population changed between $t = 2$ and $t = 6$:

$$\Delta P = P(6) - P(2) = 250 - 140 = 110.$$

The population increased by 110 people from $t = 2$ to $t = 6$. ΔP shows change in the population measured in number of people.

(b) The average **rate** of change tells us how fast was the population changing between $t = 2$ and $t = 6$ and is measured in people/year:

$$\text{Average rate of change} = \frac{\Delta P}{\Delta t} = \frac{P(6) - P(2)}{6-2} = \frac{250-140}{6-2} = \frac{110}{4} = 27.5\,\frac{\text{people}}{\text{year}}.$$

Example 1.5.5. A ball is dropped from the rooftop of a building 150 feet tall. The height of the ball above the ground, $h(t)$, in feet, t seconds after the ball is dropped is:

$$h(t) = 150 - 16t^2.$$

(a) Find the average rate of change of $h(t)$ between $t = 0$ and $t = 2$.

(b) When will the ball hit the ground?

Solution. (a) We use the definition of the average rate of change:

$$\text{Average rate of change} = \frac{\Delta h}{\Delta t} = \frac{h(2) - h(0)}{2-0} = \frac{(150 - 16 \cdot 2^2) - (150 - 16 \cdot 0^2)}{2} = \frac{86 - 150}{2} = -32\,\frac{\text{ft}}{\text{sec}}.$$

The average rate of change is $-32\,\frac{\text{ft}}{\text{sec}}$. This is, of course, the average velocity of the ball between $t = 0$ and $t = 2$. The velocity is negative because the height is decreasing.

(b) The ball hits the ground when its height is 0; that is, when $h(t) = 150 - 16t^2 = 0$. We have to solve the equation for t:

$$\begin{aligned} 150 - 16t^2 &= 0 \\ 150 &= 16t^2 \\ \frac{150}{16} &= t^2 \\ t^2 &= 9.375 \\ t &= \pm\sqrt{9.375} \end{aligned}$$

Mathematically, we get two solutions $t = \sqrt{9.375}$ and $t = -\sqrt{9.375}$. In our problem, t cannot be negative as it is the number of seconds after the ball is dropped. So the time when the ball hits the ground is $t = \sqrt{9.375} \approx 3.062$ seconds after it is dropped.

Practice Problems for Section 1.5

In Problems 1-5 given a function and an interval find the average rate of change. Round off your answers to two decimal places.

1. $f(x) = 1 + x^2$ between $x = 1$ and $x = 5$.

2. $f(x) = 1 + x^2$ between $x = -3$ and $x = 0$.

3. $g(t) = \frac{1}{t}$ between $t = 1$ and $t = 3$.

4. $p(x) = 2 - 3x$ between $x = 1$ and $x = 5$.

5. $p(x) = 2 - 3x$ between $x = -1$ and $x = 2$.

6. Let's look again at Table 1.8 with data about the world population, in billions, between the years 2010 and 2018:

Year	2010	2011	2012	2013	2014	2015	2016	2017	2018
Population	6.957	7.041	7.126	7.211	7.296	7.380	7.464	7.548	7.631

(a) What was the average rate of change of the world population between 2010 and 2012? Between 2016 and 2018? Round off your answers to four decimal places. Give units with your answers.

(b) Is there any one-year interval during the time period 2010-2018 in which the average rate of change of the population was negative?

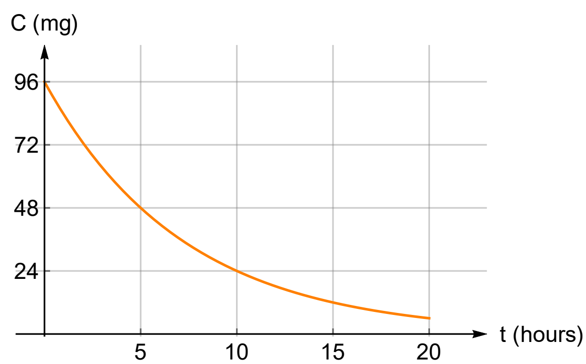
7. A ball is dropped from a cliff above a lake. The ball's height above the surface of the lake, $h(t)$, in feet, t seconds after the ball is dropped, is given by:

$$h(t) = 300 - 16t^2$$

(a) What is the average rate of change in height $h(t)$ between $t = 0$ and $t = 3$? Between $t = 1$ and $t = 4$? Give units with your answers.

(b) Are your answers in part (a) positive or negative? Explain why.

8. Let $C(t)$ be the amount of caffeine, in milligrams, remaining in the body t hours after drinking a cup of coffee. Below is the graph of the function $C(t)$:

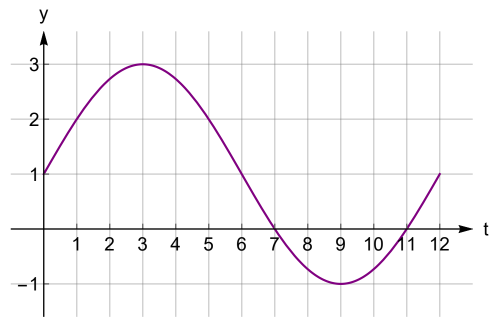


(a) Estimate the average rate of change of $C(t)$ between $t = 0$ and $t = 5$. Give units with your answer.

(b) Estimate the average rate of change of $C(t)$ between $t = 10$ and $t = 15$. Give units with your answer.

(c) Compare the magnitudes of your answers in (a) and (b). Can you explain the difference by what you see on the graph?

9. The graph of a function $y = f(t)$ is below.



Without performing any calculations, answer the following questions. Explain your answers.

(a) Is the average rate of change of $f(t)$ between $t = 0$ and $t = 3$ positive or negative?

(b) Is the average rate of change of $f(t)$ between $t = 3$ and $t = 6$ positive or negative?

10. A person's weight, W, in pounds, depends on the number of minutes m of daily exercise. Hence, $W = W(m)$ is a function of m. Below is a numerical representation of $W(m)$:

m (minutes)	0	20	40	60	80	100	120
W (pounds)	159	152	146	141	137	136	135.6

(a) Find the average rate of change of $W(m)$ between $m = 0$ and $m = 40$. Give units with your answer.

(b) Find the average rate of change of $W(m)$ between $m = 80$ and $m = 120$. Give units with your answer.

(c) Based on your answers and the data in the table, what can you say about the rate of weight loss with increasing amount of exercise?

11. Worldwide sales of passenger cars have been fluctuating between 2012 and 2019[3]:

Year	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019
Cars Sold (Millions)	55.82	57.84	60.94	63.43	65.71	66.31	69.46	70.69	68.68	64.34

(a) Calculate the average rate of change in sales of passenger cars between the years 2010 and 2017. Give units with your answer.

(b) Calculate the average rate of change in sales of passenger cars between the years 2017 and 2019. Give units with your answer.

(c) In what one-year intervals was the average rate of change negative?

[3] http://www.oica.net/wp-content/uploads/pc_sales_2019.pdf, accessed: 6/30/20

Chapter 2

Linear Functions

2.1 Basic Properties of Linear Functions

The simplest functions are linear functions. They are simple algebraically, graphically, and when a linear function appears in the context of a real-life process, the meaning of all elements of the function is clear.

Definition of a Linear Function

A function $y = f(x)$ is called a **linear function** if $f(x)$ can be written in the form:

$$f(x) = mx + b$$

where m and b are given constants. The constant m is called the **slope** of the function $y = f(x)$. The constant b is the initial value of $f(x)$ and it is also the vertical intercept of the function as $f(0) = m \cdot 0 + b = b$.

The form $f(x) = mx + b$ of a linear function is called the **slope-intercept form**.

Let's look at applied examples of linear functions to understand the practical meaning of the slope m and the initial value b. In applied examples the input and output variables will often be denoted by letters that correspond to their practical meaning rather than by x and y.

Example 2.1.1. Maya is saving money to purchase a car. She initially has 60 dollars in savings and for each hour she works at her summer job she is able to put an additional 5 dollars into savings.

If Maya works...	**Maya's savings will be...**
0 hours	5(0)+60=60 dollars
1 hour	5(1)+60=65 dollars
2 hours	5(2)+60=70 dollars
3 hours	5(3)+60=75 dollars
⋮	⋮

Table 2.1

We can see a pattern unfolding here and write a formula for the amount of money A that Maya has in savings as a function of the number of hours n that she works:

$$A = f(n) = 5n + 60$$

The function $A = f(n)$ is a linear function. Its slope is $m = 5$ and its vertical intercept is $b = 60$.

Mathematically the domain of $f(n)$ consists of all inputs n but, given the practical meaning of the function, we restrict its domain to $n \geq 0$. (Maya cannot work a negative amount of hours).

Observe that as in most of applied examples the input and output variables have units: n is measured in hours worked; A is measured in dollars. Likewise both constants $m = 5$ and $b = 60$ have units and a practical meaning. $b = 60$ dollars is the initial value of A at $n = 0$; that is, the amount of money that Maya has in savings initially. b is measured in dollars.

The slope $m = 5$ means in practical terms that Maya saves 5 dollars per each hour worked. Hence, the slope $m = 5$ is measured in dollars per hour, dollars/hour, and gives the rate at which Maya's savings A grow as the number of hours n increases. We see that Maya's savings grow at a constant rate. Let's check the average rate of change on a few intervals:

The average rate of change from $n = 0$ to $n = 1$ is $\dfrac{65 - 60}{1 - 0} = \dfrac{5}{1} = 5$ dollars per hour.

The average rate of change from $n = 1$ to $n = 2$ is $\dfrac{70 - 65}{2 - 1} = \dfrac{5}{1} = 5$ dollars per hour.

The average rate of change from $n = 2$ to $n = 3$ is $\dfrac{75 - 65}{3 - 2} = \dfrac{5}{1} = 5$ dollars per hour.

The average rate of change from $n = 0$ to $n = 3$ is $\dfrac{75 - 60}{3 - 0} = \dfrac{15}{3} = 5$ dollars per hour.

The rate of change of A with respect to n is constant and equal to 5 dollars/hour.

As the example above illustrates, for linear functions — and only for linear functions — the rate of change of the dependent variable with respect to the independent variable is constant and equal to the slope.

Slope as Rate of Change, Units of m and b

For every linear function $y = f(x) = mx + b$, the rate of change of y with respect to x is constant and equal to:

$$m \text{ (units of } y)/(\text{unit of } x).$$

b represents the initial value of the function, $b = f(0)$, and is measured in units of y.

In case you are curious, the constancy of the rate of change of a linear function follows easily from algebraic properties of linear functions. Take two distinct inputs x_1, x_2 and the corresponding outputs $y_1 = f(x_1)$, $y_2 = f(x_2)$. The average rate of change of $f(x)$ between x_1 and x_2 is:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = \frac{mx_2 + b - mx_1 - b}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m$$

Example 2.1.2. The population of a town, $P = P(t)$, in the number of people, t years after the year 2010 is given by a linear function:

$$P(t) = 150t + 4500.$$

Identify the slope and the initial value of the function. Give their units and explain their practical meaning.

Solution. From the formula for $P(t)$, we see that the slope of the function is $m = 150$ and the initial value $b = 4500$. The slope m is measured in people/year and it gives the rate of change of the population: $P(t)$ is increasing at the constant rate of 150 people/year. In practical terms, it means that each year the population increases by 150 people. $b = 4500$ is measured in the number of people and it represents the initial value of the population at $t = 0$; that is, in 2010. In 2010 the population of the town was 4500 people.

Example 2.1.3. A container holding 50 gallons of water has sprung a leak. Water is leaking out at the rate of 5 gallons/hour. Let $W(t)$ be the amount of water in the container, in gallons, t hours after the leak began. Find a formula for the function $W(t)$ and explain the practical meaning of all constants involved.

Solution. Water is leaking out at a constant rate so $W(t)$ is a linear function, $W(t) = mt + b$. The amount of water $W(t)$ in the container is *decreasing* at the rate 5 gallons/hour. Hence, the rate of change of $W(t)$ is -5 gallons/hour. That gives the slope $m = -5$ measured in gallons/hour. The initial amount of water is $W(0) = 50$ gallons, hence, $b = 50$ and its units are gallons. The formula for $W(t)$ is:

$$W(t) = -5t + 50.$$

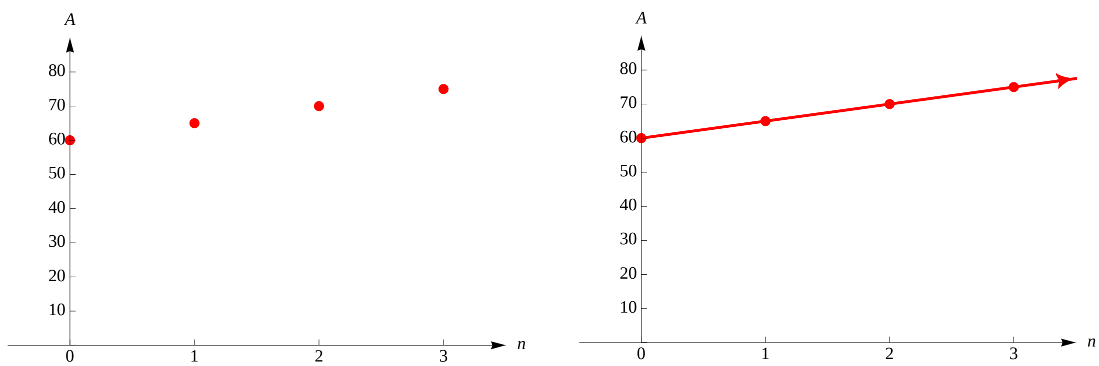
Graphs of Linear Functions

Graphs of linear functions — and only of linear functions — are straight lines.

Example 2.1.4. Consider again Maya's financial situation described in Example 2.1.1 and the function

$$A = f(n) = 5n + 60$$

that gives Maya's savings A, in dollars, as a function of the number n of hours that Maya works. This time let's draw the graph of the function $f(n)$. In Table 2.1 we have four points on the graph of $A = 5n + 60$: $(0, 60)$, $(1, 65)$, $(2, 70)$, and $(3, 75)$. We use the points to plot the graph of the function:



We see clearly that the points lie along a straight line — the graph of the function $A = 5n + 60$ is a straight line. It is the line whose slope is $m = 5$ and the vertical intercept is $b = 60$.

The Graph of a Linear Function

The graph of a linear function

$$y = f(x) = mx + b$$

is a straight line with slope m and vertical intercept b. The equation of the line in the **slope-intercept** form is $y = mx + b$.

- If the slope $m > 0$, the function $f(x)$ is increasing and its graph is climbing as x moves from left to right.
- If the slope $m < 0$, the function $f(x)$ is decreasing and its graph is falling as x moves from left to right.
- If the slope $m = 0$, the function $f(x)$ is constant, $f(x) = b$, and its graph is the horizontal line $y = b$.

Example 2.1.5. Consider the function $W(t)$ from Example 2.1.3. The function gives the amount of water in a container, $W(t)$, in gallons, t hours after the container sprung a leak:

$$W(t) = -5t + 50$$

The slope is negative, $m = -5$, so the graph is falling; it falls 5 units of W per every unit increase in t. The vertical intercept is $b = 50$ gallons. The graph of the function looks as follows:

The graph shows clearly the practical significance of vertical and horizontal intercepts. The vertical intercept, $W = 50$, gives the initial amount of water in the container. The horizontal intercept, $t = 10$, shows the time needed for all the water to leak out.

The graph of every linear function $y = f(x) = mx + b$ is a line on the xy-plane. And vice versa: every (non-vertical) line has an equation of the form $y = mx + b$. So every non-vertical line is the graph of a linear function. Sometimes people talk about linear functions and lines on the plane almost interchangeably. Despite the duality between lines and linear functions, it is important to remember that a function is a dependence between two numerical variables while a line is its visual representation.

Here is a very useful formula for calculating the slope of a linear function given two points on its graph. In other words, a formula for calculating the slope of a line through two given points.

Slope Formula

The slope m of a linear function $y = f(x)$ whose graph passes through two distinct points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{change in output}}{\text{change in input}} = \frac{\text{vertical change}}{\text{horizontal change}}$$

In particular, the formula gives the slope of a line through two given points (x_1, y_1) and (x_2, y_2).

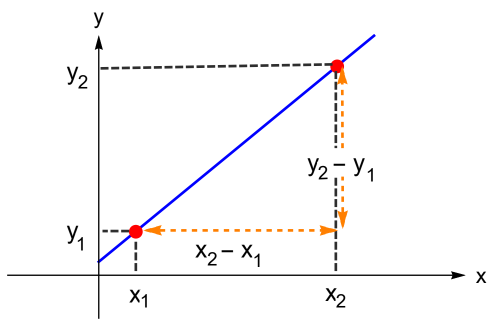
To see why the formula is true, notice that points (x_1, y_1) and (x_2, y_2) are on the graph of

$y = f(x)$ if $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Hence:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \text{the average rate of change of } f(x) \text{ between } x_1 \text{ and } x_2.$$

But the rate of change of a linear function is constant and equal to m. So the quotient is equal to m.

Graphically, the slope formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ can be illuminated as follows:



Example 2.1.6. Find a linear function $f(x) = mx + b$ such that $f(1) = 2$ and $f(3) = -2$.

Solution. We have two points on the graph of the function: $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (3, -2)$. Using the slope formula we can find m:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 2}{3 - 1} = -2.$$

So, $f(x) = -2x + b$. We can't see b right away as we don't have the value of the function at $x = 0$. But we can use either one of the two points on the graph, for example, $(1, 2)$ to set up an equation for b. We have $f(1) = 2$ so substituting $x = 1$ into $f(x) = -2x + b$ we have:

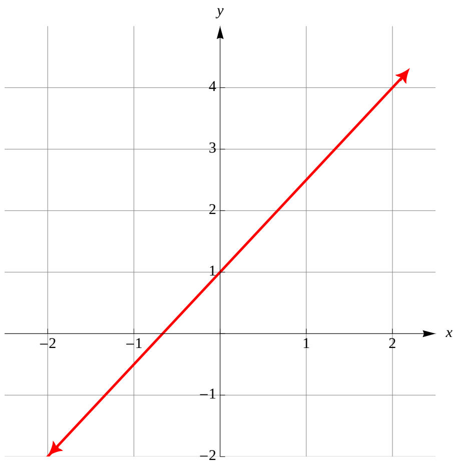
$$-2 \cdot 1 + b = 2$$

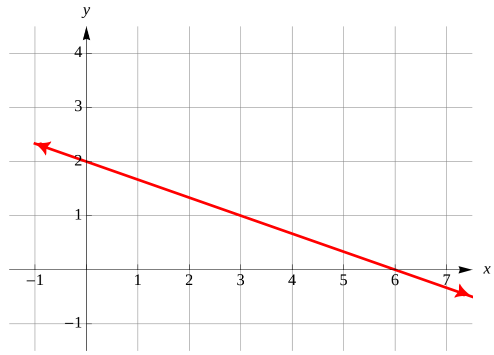
Hence:

$$b - 2 = 2$$

which gives $b = 4$. The final answer is: $f(x) = -2x + 4$. In the next section, we will learn a simpler way of solving such problems: the point-slope form of a linear function.

Example 2.1.7. Identify the slope and vertical intercept of each line. Then write an equation of the line.

(a) 

(b) 

Solution. (a) The line intersects the y-axis at the point $(0, 1)$. Hence, the vertical intercept is $b = 1$. To find the slope, we have to find two points on the graph where we can clearly read both coordinates so we can use the slope formula. One such point is $(x_1, y_1) = (0, 1)$; the second is $(x_2, y_2) = (2, 4)$. We calculate the slope m:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{2 - 0} = \frac{3}{2}$$

The equation of the line is $y = \dfrac{3}{2}x + 1$. The corresponding linear function is, of course, $f(x) = \frac{3}{2}x + 1$.

(b) Again we need two points on the line with clearly visible coordinates. Two such points are $(x_1, y_1) = (0, 2)$ and $(x_2, y_2) = (6, 0)$. We use the slope formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 2}{6 - 0} = \frac{-2}{6} = -\frac{1}{3}.$$

The vertical intercept is $b = 2$. The equation of the line is $y = -\dfrac{1}{3}x + 2$. The corresponding linear function is $f(x) = -\frac{1}{3}x + 2$.

Example 2.1.8. Find the slope of the line passing through the two given points.

(a) $(-3, -1)$ and $(-2, 4)$ (b) $(5, 0)$ and $(-6, 0)$ (c) $(1/2, 6)$ and $(1, -1/3)$.

Solution. (a) If $(x_1, y_1) = (-3, -1)$ and $(x_2, y_2) = (-2, 4)$ then the slope is given by

$$m = \frac{4 - (-1)}{-2 - (-3)} = \frac{4 + 1}{-2 + 3} = \frac{5}{1} = 5.$$

(b) If $(x_1, y_1) = (5, 0)$ and $(x_2, y_2) = (-6, 0)$ then the slope is given by

$$m = \frac{0 - 0}{-6 - 5} = \frac{0}{-11} = 0.$$

(c) If $(x_1, y_1) = (1/2, 6)$ and $(x_2, y_2) = (1, -1/3)$ then the slope is given by

$$m = \frac{-1/3 - 6}{1 - 1/2} = \frac{-1/3 - 18/3}{2/2 - 1/2} = \frac{-19/3}{1/2} = -\frac{19}{3} \cdot \frac{2}{1} = -\frac{38}{3}.$$

Practice Problems for Section 2.1

In Problems 1-4 rewrite the given linear function in the slope-intercept form. Identify the slope and the vertical intercept of each function.

1. $f(x) = 5x - 2(x + 3)$
2. $g(x) = 4(2x - 1) + 2$
3. $h(t) = \dfrac{18t - 5}{2}$
4. $m(x) = 2(x - 1) + 2$

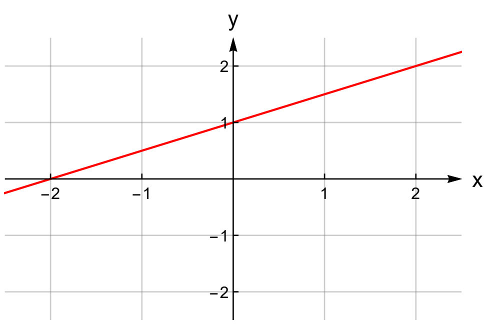
In Problems 5-10 find the slope of the line passing through the given points.

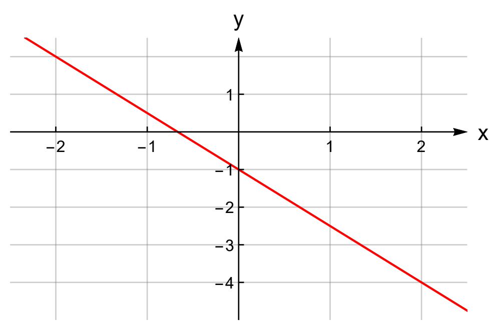
5. $(0, 10)$ and $(3, 16)$
6. $(6, -4)$ and $(-15, -1)$
7. $(-0.7, 7.2)$ and $(-0.5, 12.5)$
8. $(\frac{1}{3}, 1)$ and $(-1, \frac{1}{2})$
9. $(3, -5)$ and $(8, -5)$
10. $(0, 0)$ and $(2, 2)$

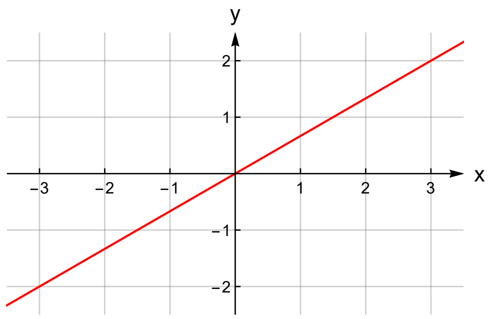
In Problems 11-14 find the slope of a linear function given its outputs for two inputs.

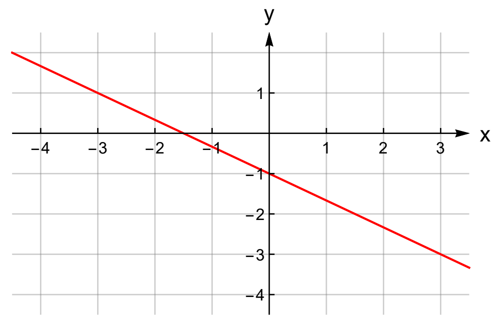
11. $f(1) = -2$ and $f(3) = 1$
12. $g(0.5) = 0$ and $g(2) = -2$
13. $h(-1) = -2$ and $h(1) = 2$
14. $w(2) = -3$ and $w(6) = -3$

In Problems 15-18 Identify the slope and vertical intercept of each line. Then write an equation of the line.

15. 

16. 

17. 

18. 

19. Five linear functions (a)-(e) are given below. Match each function with its graph from the figure to the right.

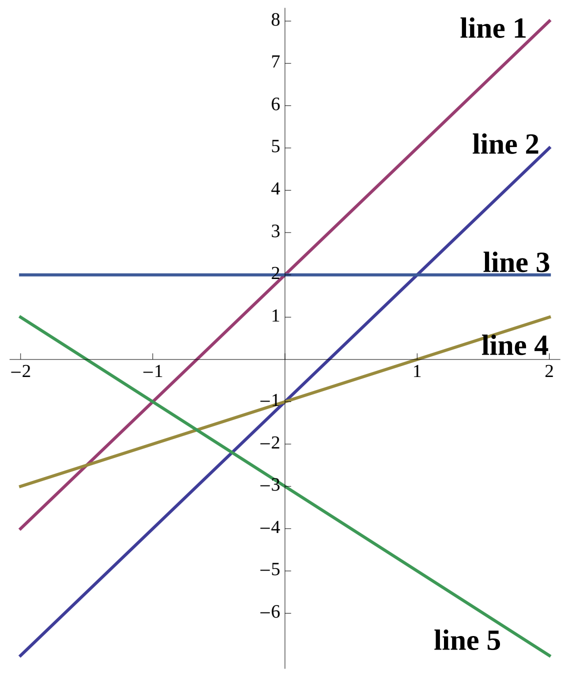
(a) $y = -2x - 3$

(b) $y = 3x - 1$

(c) $y = 3x + 2$

(d) $y = x - 1$

(e) $y = 2$



20. During the 2019-2020 academic year, a senior lecturer at URI earned a base salary of \$55,683 for a standard teaching load. For each credit taught in excess of the standard load, \$1,567 were added to his salary. Write a linear equation expressing the total amount A that the senior lecturer earned for teaching n credits in excess of the standard workload during the 2019-2020 academic year.

21. The elevation in feet, $E(t)$, of a hiker t minutes after beginning her hike is given by $E(t) = 35t + 1350$.

(a) Write a complete sentence explaining the practical meaning of the vertical intercept of this linear function. Include units in your answer.

(b) Write a complete sentence explaining the practical meaning of the slope of this linear function. Include units in your answer.

22. The distance in miles from the finish line, $D(t)$, of a bicyclist t hours after beginning a race is given by $D(t) = 50 - 25t$.

  (a) Write a complete sentence explaining the practical meaning of the vertical intercept of this linear function. Include units in your answer.
  (b) Write a complete sentence explaining the practical meaning of the slope of this linear function. Include units in your answer.

2.2 Working with Linear Functions, Linear Equations

Point-Slope Form of Lines and Linear Functions

How to find an equation of a line given its slope and a point through which the line passes? How to find a formula for a linear function given its slope and one point on its graph? Use the point-slope form.

Point-Slope Form

- The point-slope form of the equation of a line with slope m passing through (x_0, y_0) is:
$$y - y_0 = m(x - x_0).$$
- A linear function $y = f(x)$ with slope m for which $f(x_0) = y_0$ can be written in the point-slope form as:
$$f(x) = y_0 + m(x - x_0).$$
Note that $f(x_0) = y_0$ means that the point (x_0, y_0) is on the graph of the function $y = f(x)$.

The point-slope form is very useful for finding formulas for linear functions and equations of lines.

Example 2.2.1. Find an equation of the line passing through the given points. Rewrite your equation in the slope-intercept form.

(a) $(2, -1)$ and $(4, 5)$ (b) $(-1, -2)$ and $(1, 3)$ (c) $(0, 0)$ and $(1, 1.5)$

Solution. In all parts (a)-(c), we follow the same steps. We use the two given points to find the slope. Having the value of the slope, we pick either one of the two points as (x_0, y_0) and write an equation in the point-slope form. Finally, we rewrite the equation in the slope-intercept form.

(a) $m = \dfrac{5 - (-1)}{4 - 2} = \dfrac{6}{2} = 3$. We have the slope $m = 3$. Now we need a point on the line. We have two such points. Let's choose one, say $(x_0, y_0) = (2, -1)$, and use the point-slope form. The equation of the line is $y-(-1)=3(x-2)$ which simplifies to $y+1=3x-6$. By adding -1 to both sides, we obtain the slope-intercept form $y=3x-7$.

(b) We use the points $(-1,-2)$ and $(1,3)$ to find the slope: $m=\dfrac{3-(-2)}{1-(-1)}=\dfrac{5}{2}$. Take $(x_0,y_0)=(1,3)$. The equation of the line is $y-3=\dfrac{5}{2}(x-1)$ which simplifies to $y-3=\dfrac{5}{2}x-\dfrac{5}{2}$. We add $\dfrac{6}{2}$ to both sides and obtain $y=\dfrac{5}{2}x+\dfrac{1}{2}$.

(c) $m=\dfrac{1.5-0}{1-0}=\dfrac{1.5}{1}=1.5$. Take $(x_0,y_0)=(0,0)$. The point-slope form gives $y-0=1.5(x-0)$ which simplifies to the slope-intercept form $y=1.5x$. As we see, the y-intercept of the line is 0. Hence the line passes through the origin $(0,0)$.

Example 2.2.2. Let $y=f(x)$ be a linear function such that $f(1)=-2$ and $f(3)=-5$. Find a formula for the function in the slope-intercept form.

Solution. The values of f for two inputs give us two points on the graph of the function: $(1,-2)$ and $(3,-5)$. We can calculate the slope: $m=\dfrac{-5-(-2)}{3-1}=-\dfrac{3}{2}$. Take $(x_0,y_0)=(1,-2)$ and write the point-slope formula for $f(x)$:

$$f(x)=-2+(-\frac{3}{2})(x-1).$$

The formula simplifies to the slope-intercept form:

$$f(x)=-\frac{3}{2}x-\frac{1}{2}.$$

Linear Functions and Linear Equations

Example 2.2.3. Mr. Bell is driving through Nevada to Reno at a constant speed of 40 mph. Let $D(t)$ be his distance from Reno, in miles, t hours after he began driving. At $t=2$, Mr. Bell checks his GPS to find that his distance from Reno is 150 miles.

(a) Find a formula for the function $D(t)$.

(b) How far from Reno was Mr. Bell when he began driving?

(c) How long from the moment he began driving will it take him to reach Reno?

Solution. (a) At first it seems that we don't have enough information to find a formula for $D(t)$. We only have the value of the function at one point: $D(2)$=150. However, we also know that Mr. Bell is driving at the constant speed of 40 mph. Hence, his distance from Reno is decreasing at the constant rate of 40 miles per hour. In other words, $D(t)$ is changing at the constant rate of -40

miles per hour. The rate of change of $D(t)$ is constant so the function $D(t)$ is linear. We have the constant rate of change of the function $D(t)$ so we have its slope: $m = -40$. We also have a point on the graph $(2, 150)$. Given the slope and a point, we can write the point-slope form for $D(t)$:

$$D(t) = 150 - 40(t - 2).$$

We can rewrite the formula in the slope-intercept form using simple algebra:
$D(t) = 150 - 40(t - 2) = 150 - 40t + 80 = 230 - 40t$. The slope-intercept form is:

$$D(t) = 230 - 40t.$$

(b) Mr. Bell began driving at $t = 0$ when, according to the formula, his distance from Reno was $D(0) = 230$ miles.

(c) Mr. Bell will reach Reno when his distance from Reno is 0; that is, at t such that:

$$D(t) = 230 - 40t = 0.$$

To find such t we have to solve the equation:

$$230 - 40t = 0.$$

An equation in which the unknown appears in the first power is called a **linear equation**. Linear equations appear very often in the context of linear functions and it is a good opportunity to review solving such equations. To solve our equation, we apply the standard methods of adding or subtracting a number or an expression from both sides, multiplying or dividing both sides by a non-zero number or expression, grouping similar terms, etc. To solve our equation, we subtract 230 from both sides and obtain:

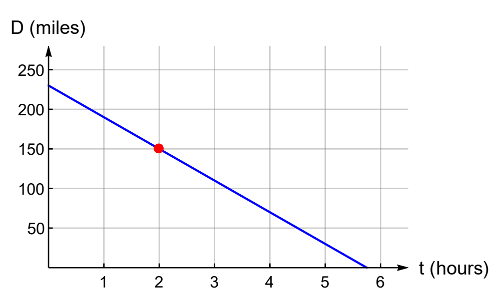
$$-40t = -230.$$

Now we divide both sides by -40 and obtain:

$$t = \frac{23}{4} = 5.75.$$

It takes Mr. Bell 5.75 hours total to reach Reno. 5.75 hours translates into 5 hours and 45 minutes.

Observe that to answer part (c), we had to find the **horizontal intercept** of the function $D(t)$. In applied problems horizontal intercepts usually have important practical meaning just as vertical intercepts do. Finding the horizontal intercept of a linear function from its formula usually requires solving a linear equation. Here is the graph of the function $D(t)$ that illustrates graphically the meaning of the horizontal intercept:



Example 2.2.4. During the first ten days after hatching, a chick gains weight fast at the rate of 18.9 grams/day. Let $w(d)$ be the weight of the chick d days after hatching. Three days after hatching the chick weighs 100.6 grams.

(a) Find a formula for the function $w(d)$.

(b) What was the weight of the chick when it hatched?

(c) How many days after it hatched will the chick weigh 120 grams?

Solution. (a) The function $w(d)$ changes at a constant rate so it is linear. The slope m is equal to the constant rate of change so $m = 18.9$. We have the value of the function at $d = 3$: $w(3) = 100.6$. Hence, the point $(d_0, w_0) = (3, 100.6)$ is on the graph of the function $w(d)$. We use the point-slope form and get:

$$w(d) = 100.6 + 18.9(d - 3).$$

We rewrite $w(d)$ in the slope-intercept form: $w(d) = 100.6 + 18.9d - 18.9 \cdot 3 = 43.9 + 18.9d$. In the slope-intercept form the function is:

$$w(d) = 43.9 + 18.9d.$$

(b) The weight of the chick when it hatched was $w(0) = 43.9 + 18.9 \cdot 0 = 43.9 \approx 44$ grams.

(c) The weight of the chick will reach 120 grams for d for which $w(d) = 120$. That is, for d that satisfies the linear equation:

$$43.9 + 18.9d = 120.$$

To solve the equation, we subtract 43.9 from both sides and then divide both sides by 18.9:

$$d = \frac{120 - 43.9}{18.9} = 4.026 \approx 4.$$

It will take approximately 4 days for the chick to reach the weight of 120 grams.

Example 2.2.5. The populations of two twin towns — one in the Netherlands and one in France – are $N(t)$ and $F(t)$, respectively, where t is the number of years since a twin partnership was established. Both populations are measured in the number of people. Both populations grow linearly according to the formulas:

$$N(t) = 8200 + 400t \qquad \text{and} \qquad F(t) = 6500 + 600t.$$

(a) Which town had a larger population initially when a twin partnership was established?

(b) The population of which of the two towns grows faster and at what rate does it grow?

(c) Will the two populations ever be equal and if yes, when?

Solution. (a) The initial population of the Dutch town is $N(0) = 8200$ people, of the French town $F(0) = 6500$. Initially, the population of the Dutch town was larger.

(b) It is the slope of each function that gives its rate of increase. The population of the Dutch town grows at 400 people/year. The population of the French town grows faster at 600 people/year.

(c) At the beginning, the population of the French town is lower than the population of the Dutch town. Then, however, the population of the French town grows faster. Is there a time when the populations are equal? That would be the time t for which:

$$F(t) = N(t).$$

We use the formulas for the population functions to set up an equation for t:

$$8200 + 400t = 6500 + 600t.$$

This is a linear equation for t. To solve it, we move all the constants to one side and all the terms containing t to the other. In other words, we subtract 6500 from both sides and then we subtract 400t from both sides:

$$8200 - 6500 = 600t - 400t.$$

Combining like terms gives:

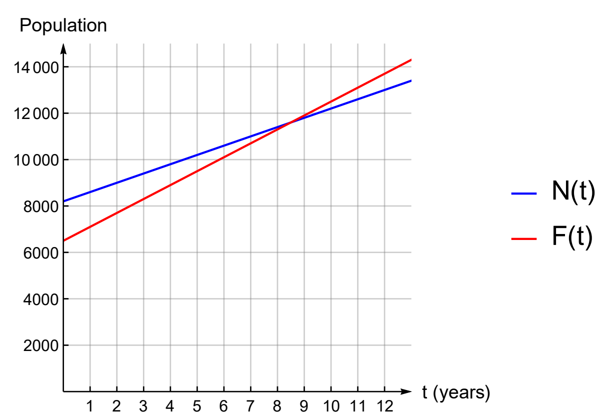
$$1700 = 200t.$$

We divide both sides by 200 and obtain:

$$t = 8.5.$$

After 8.5 years the populations of both towns are equal. After that the population of the French town will always be larger as it is increasing faster.

It is worthwhile to look at the graphs of both functions:



We see how a larger slope produces a steeper line; we see the meaning of the vertical intercept as the initial value.

Example 2.2.6. The value $V(t)$ of a car, in dollars, t years after the car was purchased is given by:

$$V(t) = 15000 - 1250t.$$

Find the vertical intercept and the horizontal intercept of the function $V(t)$ and explain their meaning in practical terms.

Solution. The vertical intercept is the value of the function at $t = 0$:

$$V(0) = 15000 - 1250 \cdot 0 = 15000.$$

In practical terms, it is the value of the car, in dollars, at $t = 0$; that is, it is the purchase value of the car. To find the horizontal intercept, we have to find t for which:

$$V(t) = 0.$$

Hence, we have to solve a linear equation:

$$15000 - 1250t = 0.$$

We subtract 15000 from both sides:

$$-1250t = -15000$$

and then divide both sides by -1250:

$$t = \frac{-15000}{-1250} = 12.$$

The value of the car is 0 at $t = 12$; that is, after 12 years the car is worth nothing. Here is the graph of the function $V(t) = 15000 - 1250t$ with intercepts clearly visible:

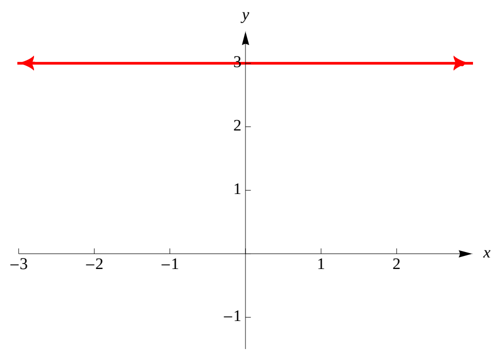
Note that practically speaking, the domain of the function $V(t)$ should be limited to $0 \leq t \leq 12$. It doesn't make much sense to talk about the value of the car being negative. Mathematically, the formula $V(t) = 15000 - 1250t$ gives an output for every input t so, mathematically, the domain consists of all t.

Constant Functions, Horizontal and Vertical Lines

Consider a linear function with slope $m = 0$. For example, take

$$y = f(x) = 0 \cdot x + 3.$$

For every x, the value of the function is the same: $f(x) = 3$. We say that the function $f(x)$ is **constant** and it is constantly equal to 3. It is not a surprise that the function is constant given that the rate of change of the function is $m = 0$ — the function doesn't change. The graph of the function consists of all points on the plane for which $y = 3$; that is, the graph is the horizontal line through the y-value 3 as depicted below.

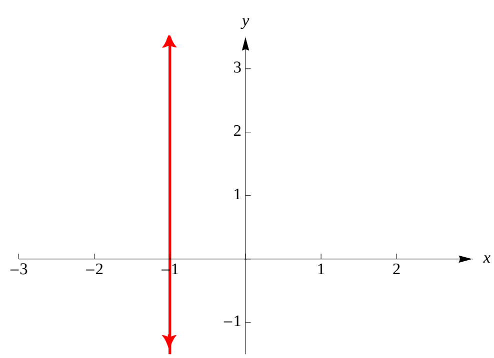


In general, the graph of a linear function with slope $m = 0$ and the vertical intercept b:

$$y = f(x) = 0 \cdot x + b$$

is a horizontal line through the y-value b.

So horizontal lines on the plane are graphs of constant functions. How about vertical lines? Vertical lines, that is, lines parallel to the y-axis are not graphs of functions. A vertical line consists of points for which the x coordinate is constant. Take for example $x = -1$:



The line is not the graph of any function $y = f(x)$ as to one input $x = -1$ there is more than one output — infinitely many outputs, in fact.

Vertical lines are not graphs of functions. It may be worth noticing that slopes of vertical lines are undefined. Take two points on the line $x = -1$, say $(-1, 0)$ and $(-1, 1)$. The slope formula gives:

$$m = \frac{1-0}{-1-(-1)} = \frac{1}{-1+1} = \frac{1}{0}$$

which is undefined due to division by 0.

Practice Problems for Section 2.2

In Problems 1-8 write the equation of the line satisfying each set of conditions in slope-intercept form.

1. passing through $(2, 6)$ and $(1, 8)$
2. passing through $(-4, 7)$ and $(-10, -8)$
3. with slope 2.1 passing through $(0.1, 0)$
4. with slope 6 passing through $(-5, 2)$
5. with slope $-\frac{1}{2}$ passing through $(\frac{2}{5}, -1)$
6. passing through $(1, 3)$ and $(-7, 3)$
7. passing through $(5, -5)$ and $(5, 2)$
8. with slope 0 and passing through $(-3, -1)$

In Problems 9-16 write the formula in slope-intercept form for a function $f(x)$ satisfying each set of conditions.

9. $f(2) = -5$ and $f(4) = 0$
10. $f(1) = -2$ and $f(3) = -5$
11. $f(x)$ has slope 2.1 and its graph passes through $(0.1, 0)$
12. $f(x)$ has slope -1 and its graph passes through $(-1, 2)$
13. the graph of $f(x)$ passes through $(-1, 3)$ and $(-7, -2)$
14. the slope of $f(x)$ is 3 and its horizontal intercept is -2
15. the slope of $f(x)$ is 1 and its vertical intercept is 2
16. the vertical intercept of $f(x)$ is 3 and its horizontal intercept is 7

17. Which of the following lines is NOT the graph of a linear function:

    (a) $y = -2x - 3x + 1$ (b) $y = -5$ (c) $x = 7$

18. The distance in miles from the finish line, $D(t)$, of a bicyclist t hours after beginning a race is given by $D(t) = 60 - 20t$

    (a) Write a complete sentence explaining the practical meaning of the vertical intercept of this linear function. Include units in your answer.

(b) Write a complete sentence explaining the practical meaning of the horizontal intercept of this linear function. Include units in your answer.

(c) Write a complete sentence explaining the practical meaning of the slope of this linear function. Include units in your answer.

19. The value of an antique lamp, $V(t)$, in dollars, t years after its purchase is given by

$$V(t) = 1900 + 250t.$$

(a) What was the purchase price of the lamp?

(b) When will the value of the lamp reach \$4000?

20. Two mobile phone companies sell an international roaming plan. Company A charges a \$75 dollar fixed monthly fee and \$0.20 per minute for talk. Company B charges a \$90 dollar fixed monthly fee and \$0.10 per minute for talk.

(a) Write the linear functions $A(m)$ and $B(m)$ which give the monthly cost charged by Company A and Company B, respectively, with m minutes spent talking on the phone.

(b) For what value of m is the cost the same with either company?

(c) Which company gives you a better deal if you plan to talk for hours?

In Problems 21-24 solve the given linear equation. If there is no solutions or infinitely many solutions, say so.

21. $0.1x - (2x - 5) = 2.5$

22. $\dfrac{-2(x-3)}{5} = x + 1$

23. $3x - 7 = 5 + 3x$

24. $2x + 2 = 4 + 2(x - 1)$

2.3 Modeling: When is a Numerically Given Function Linear?

How do we recognize that a function given numerically — through a table of values — is linear? The key is to remember that a function is linear if it changes at a constant rate. How do we check that a numerically given function changes at a constant rate? Let's look at an example.

Example 2.3.1. A biologist studies in her laboratory the growth of the larvae of a certain insect species during the last instar[1]. Her team takes weight measurements of the larvae every 6 hours for 48 hours beginning with $t = 0$. Let $W(t)$ denote the average larval weight, in grams, at time t. Here are the recorded measurements rounded off to two decimal places:

t (hours)	0	6	12	18	24	30	36	42	48
$W(t)$ (grams)	7.00	7.45	7.90	8.35	8.80	9.25	9.70	10.15	10.60

[1] http://www.biology.arizona.edu/biomath/tutorials/Linear/LinearModels.html, accessed: 6/12/2020

Can the growth of the larvae be modeled by a linear function $W(t) = mt + b$?

Solution. The answer is: Yes, if the function $W(t)$ changes at a constant rate. A function changes at a constant rate if to equal changes in the independent variable there correspond equal changes in the dependent variable. Let's look at the changes in $W(t)$ over each of the 6-hour intervals. In each interval the change in t is $\Delta t = 6$ hours.

Below, we calculate the corresponding changes in W, ΔW, in each of the 6-hour intervals:

$$W(6) - W(0) = 7.45 - 7.00 = 0.45 \text{ grams;}$$

$$W(12) - W(6) = 7.90 - 7.45 = 0.45 \text{ grams;}$$

$$W(18) - W(12) = 8.35 - 7.90 = 0.45 \text{ grams;}$$

$$W(24) - W(18) = 8.80 - 8.35 = 0.45 \text{ grams;}$$

$$W(30) - W(24) = 9.25 - 8.80 = 0.45 \text{ grams;}$$

$$W(36) - W(30) = 9.70 - 9.25 = 0.45 \text{ grams;}$$

$$W(42) - W(36) = 10.15 - 9.70 = 0.45 \text{ grams;}$$

$$W(48) - W(42) = 10.60 - 10.15 = 0.45 \text{ grams.}$$

All changes in W corresponding to changes $\Delta t = 6$ in t are $\Delta W = 0.45$. That means that the function $W(t)$ changes at a constant rate. Indeed, on every one of the 6-hour intervals the average rate of change in $W(t)$ is:

$$\frac{\Delta W}{\Delta t} = \frac{0.45}{6} = 0.075 \ \frac{\text{grams}}{\text{hour}}$$

Since the function $W(t)$ changes at a constant rate, the function is linear and can be represented as $W(t) = mt + b$ for some constants m and b. The vertical intercept b is the value of the function at $t = 0$. We have that value in the table:

$$b = W(0) = 7.00 \text{ grams}$$

The slope is the constant rate of change which we have just calculated:

$$m = \frac{\Delta W}{\Delta t} = \frac{0.45}{6} = 0.075 \ \frac{\text{grams}}{\text{hour}}$$

Hence:

$$W(t) = 0.075t + 7.00.$$

We found a mathematical model for the growth of the larvae.

Example 2.3.2. Decide which of the following tables represents a linear function. For each table that does represent a linear function, find a formula for that function.

(a)

t	2	4	6	8	10	12
$f(t)$	6.0	4.6	3.2	1.8	0.4	-1.0

(b)

x	0	3	6	9	12	15
$g(x)$	6	7.5	9.5	11.5	15.5	20

(c)

x	0	1	2	3	4	5
$h(x)$	3	4.5	6	7.5	9	10.5

Solution. (a) The changes Δt in t between any two consecutive points in the table are equal:

$$4 - 2 = 6 - 4 = 8 - 6 = 10 - 8 = 12 - 10 = 2 = \Delta t$$

We have to check if the changes, Δf between any two consecutive values in the table are equal:

$$\Delta f = f(4) - f(2) = 4.6 - 6.0 = -1.4;$$
$$\Delta f = f(6) - f(4) = 3.2 - 4.6 = -1.4;$$
$$\Delta f = f(8) - f(6) = 1.8 - 3.2 = -1.4;$$
$$\Delta f = f(10) - f(8) = 0.4 - 1.8 = -1.4;$$
$$\Delta f = f(12) - f(10) = -1.0 - 0.4 = -1.4.$$

Yes. The changes in f corresponding to equal changes in t are all equal: $\Delta f = -1.4$. Thus, the function $f(t)$ is linear; that is: $f(t) = mt + b$ for some constants m and b.

To find the slope m, we take the ratio: the change in f divided by the corresponding change in t:

$$m = \frac{\Delta f}{\Delta t} = \frac{-1.4}{2} = -0.7$$

Hence: $f(t) = -0.7t + b$. We still have to find b.

In the previous example, we could easily find the vertical intercept b as we were given the value of the function at 0. The table for $f(t)$ does not directly give us the value $f(0)$. The table gives us outputs for six other inputs t. In other words, it provides us with six points on the graph of $f(t)$. We can take anyone of those points and set up an equation for b. Take the point $(2, 6.0) = (2, 6)$ from the table. We have $f(2) = 6$ so

$$-0.7 \cdot 2 + b = 6$$

Solving for b, we obtain:

$$b = 6 + 1.4 = 7.4.$$

The final formula for the linear function $f(t)$ is:

$$f(t) = -0.7t + 7.4.$$

(b) Values of x are equally spaced: $3-0=6-3=9-6=12-9=15-12=3=\Delta x$. We have to check if all the corresponding changes in Δg are equal.

$$\Delta g = 7.5 - 6 = 1.5;$$

$$\Delta g = 9.5 - 7.5 = 2$$

$$\vdots$$

We don't have to go any further. The first two changes Δg are not equal. The function $g(x)$ is not linear. Just out of curiosity, let's look at the rest of changes in g:

$$\Delta g = 11.5 - 9.5 = 3;$$

$$\Delta g = 15.5.5 - 11.5 = 4;$$

$$\Delta g = 20 - 15.5 = 4.5.$$

The changes in g corresponding to equal changes in x are getting larger and larger. That means that the function $g(x)$ increases faster and faster — by more and more at each next step. $g(x)$ increases at an increasing rate unlike a linear function that increases or decreases at a constant rate.

(c) All changes in x are equal: $\Delta x = 1$ — at every step x increases by 1. The changes in h are also equal: at every step $h(x)$ increases by 1.5. So at every step, $\Delta h = 1.5$. Hence, $h(x)$ is a linear function, $h(x) = mx + b$. We find the slope:

$$m = \frac{\Delta h}{\Delta x} = \frac{1.5}{1} = 1.5.$$

We are given the value of h at 0: $h(0) = 3$. Hence, $b = 3$. The formula for $h(x)$ is:

$$h(x) = 1.5x + 3.$$

Example 2.3.3. A company purchased a computer system for $20,300. The company accountant decided to depreciate the item over 5 years of its "useful life" for tax purposes. The depreciated value of the system, $V(t)$, in dollars, reported to the IRS t years after the purchase (the so-called "carrying value") is given by:

t (years after purchase)	0	1	2	3	4	5
$V(t)$ (depreciated system value)	20300	17240	14180	11120	8060	5000

What is the amount of depreciation over each of the five years? The company can claim that amount as a deduction. What is the formula for the carrying value $V(t)$?

Solution. Let's look at the amount of depreciation each year:

$$\Delta V = 17240 - 20300 = -3060;$$

$$\Delta V = 14180 - 17240 = -3060;$$

$$\Delta V = 11120 - 14180 = -3060;$$

$$\Delta V = 8060 - 11120 = -3060;$$

$$\Delta V = 5000 - 8060 = -3060.$$

Each year, the computer system is depreciated by the same amount of \$3060. Hence, $V(t)$ is a linear function:

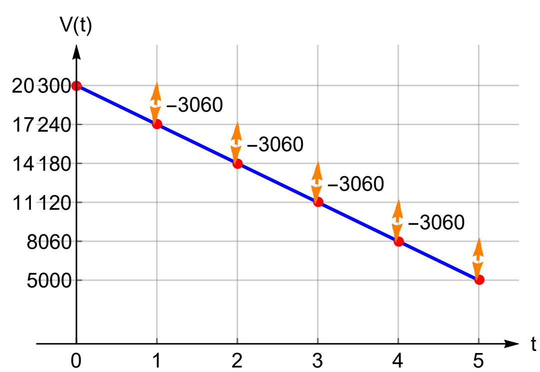
$$V(t) = 20300 - 3060t.$$

This depreciation method is called the "straight-line depreciation"[2]. The 5 years of useful life is prescribed by the IRS for each type of assets. The value at the end of the asset's useful life, called the "salvage value", can be found in accounting books.

In the straight-line appreciation method, the accountant calculates the fixed annual depreciation amount by taking the difference between the purchase cost and the salvage value and dividing by the number of years of useful life of the asset:

$$\frac{20300 - 5000}{5} = 3060.$$

The amount by which the carrying amount is changing each year and the slope of our line is then -3060. The graph of depreciated value $V(t)$ for $0 \le t \le 5$ is a straight line:



Practice Problems for Section 2.3

In problems 1-5 decide if the given table could represent a linear function and explain your reasoning. For each table that could represent a linear function, find a formula for that function.

[2] https://www.thehartford.com/business-insurance/strategy/depreciating-assets/straight-line, accessed: 6/12/20

1.

x	0	1	2	3
y	12.6	9.3	6.0	2.7

2.

t	-1	1	3	5
$f(t)$	0.1	0.4	0.7	1.0

3.

n	0	1.5	3	4.5	6
$g(n)$	12	18	12	6	12

4.

t	0	2	4	6	8
$h(t)$	6.4	1.6	0.4	0.1	0.025

5.

z	-2	1	4	7
$w(z)$	6	4.8	3.6	2.4

6. The table below shows for each temperature F in degrees Fahrenheit the corresponding approximate temperature C in degrees Celsius. Is C a linear function of F? If it is, find a formula for C as a function of F.

F (degrees Fahrenheit)	32	42	52	62
C (degrees Celsius)	0	5.55	11.12	16.67

7. The height of a human individual can be estimated by the length of the femur, as shown for males in the following table.

L (length of femur in cm)	42	45	48	51
H (height of human male in cm)	162.97	169.93	176.89	183.85

(a) Use the information in this table to find a possible formula for the height of a human male H as a function of his femur length L.

(b) What would the approximate height of a human male with femur length 46 cm be?

(c) Fernando is 175.4 cm tall. What would you expect the approximate length of his femur to be?

8. The weight of water above a scuba diver as well as the air above the diver exerts pressure on their bodies. The pressure the diver experiences at sea level is 14.7 PSI (pounds per square inch), and this pressure increases by 0.4 PSI per each foot of depth.

(a) Write a linear equation expressing the pressure P on a diver at a depth of d feet below sea level.

(b) The deepest a recreational scuba diver typically dives is 130 feet. What is the pressure on a diver at this depth?

9. A hiker is at is at a trailhead about to climb a mountain. The temperature at the trailhead is 65°F. According to the standard atmosphere model[3], the temperature drops by 0.00356°F per each 1 foot increase in altitude. Let $T = T(h)$ be the temperature, in °F, h feet above the hiker.

(a) Write a formula for the function $T(h)$.

(b) Find the temperature at the mountaintop that is 6210 feet above the hiker.

(c) Is the function $T(h)$ increasing or decreasing?

[3] https://www.grc.nasa.gov/WWW/K-12/airplane/atmos.html, accessed: 6/25/20

10. A man comes to a gym to exercise. After t minutes on a treadmill, his pulse (heart rate), H, in beats per minute, is:

t (minutes)	0	3	6	9	12	15
H (bpm)	85	87	89	91	93	95

(a) Is the function $H(t)$ linear? If yes, find a formula for H(t).

(b) Give units of the slope and the vertical intercept of $H(t)$ and explain their meaning in practical terms.

Chapter 3

Quadratic Functions

3.1 Introduction to Quadratic Functions

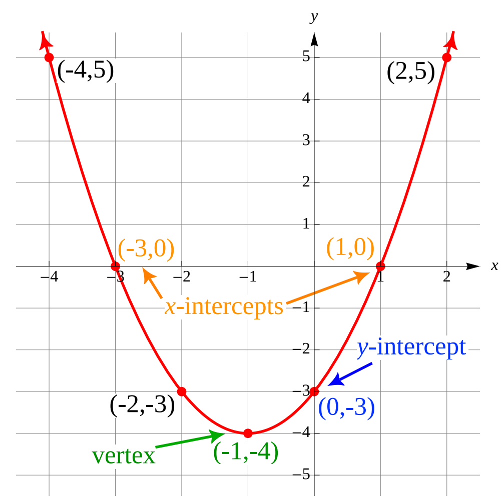
Consider the function

$$f(x) = x^2 + 2x - 3.$$

A table of values and a graph of this function are shown below.

x	$y = f(x)$	point
-4	$(-4)^2 + 2(-4) - 3 = 5$	$(-4, 5)$
-3	$(-3)^2 + 2(-3) - 3 = 0$	$(-3, 0)$
-2	$(-2)^2 + 2(-2) - 3 = -3$	$(-2, -3)$
-1	$(-1)^2 + 2(-1) - 3 = -4$	$(-1, -4)$
0	$(0)^2 + 2(0) - 3 = -3$	$(0, -3)$
1	$(1)^2 + 2(1) - 3 = 0$	$(1, 0)$
2	$(2)^2 + 2(2) - 3 = 5$	$(2, 5)$

Table of values for $f(x) = x^2 + 2x - 3$.



Graph of $f(x) = x^2 + 2x - 3$.

It can be seen that the graph of this function has a "U"-shape. It has two x-intercepts, $x = -3$ and $x = 1$, located at the points $(-3, 0)$ and $(1, 0)$ respectively, and a y-intercept $y = -3$ at $(0, -3)$. Another special point on this graph is the place were it "turns around", referred to as its **vertex**. This is located at $(-1, -4)$.

The function $f(x) = x^2 + 2x - 3$ is an illustration of a **quadratic function**.

Definition of and Standard Form for a Quadratic Function

A function that can be written in the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are real numbers with $a \neq 0$ is called a **quadratic function**.

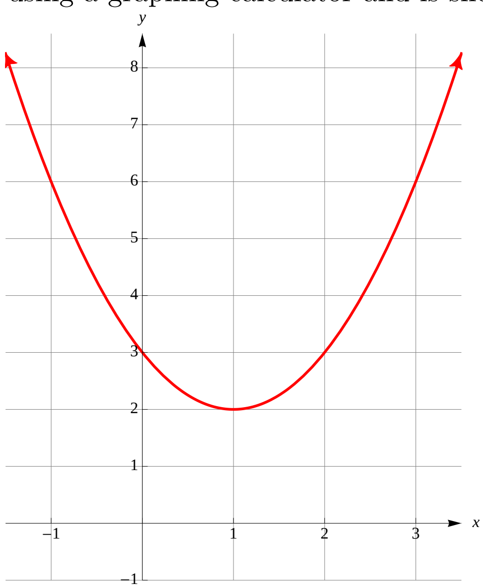
The form $f(x) = ax^2 + bx + c$ is referred to as the **standard form** for a quadratic function.

The graph of a quadratic function is "U"-shaped and called a **parabola**. If a is positive the "U" will open up and if a is negative the "U" will open down. The lowest point on a parabola that opens up (or the highest point on a parabola that opens down) is referred to as its **vertex**. Note that a parabola is symmetric about the vertical line passing through its vertex.

Example 3.1.1. Use the graph of each quadratic function to identify its intercept(s) and vertex.

(a) $f(x) = x^2 - 2x + 3$ (b) $g(x) = -2x^2 - 8x - 3$ (c) $h(x) = x^2 - 4x + 4$

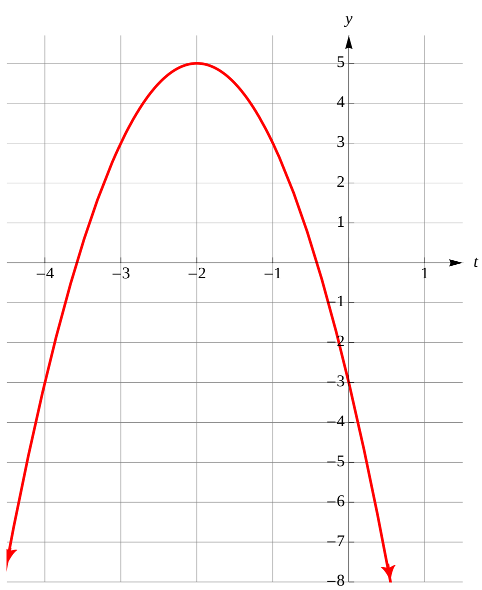
Solution. (a) The graph of this quadratic function can be obtained by making a table of values or by using a graphing calculator and is shown below.



Graph of $f(x) = x^2 - 2x + 3$.

The y-intercept is where the graph crosses the y-axis; this occurs at $y = 3$. There are no x-intercepts, since the graph is located entirely above the x-axis. The vertex is the lowest point on the graph since this is an upward facing parabola; hence the vertex is $(1, 2)$.

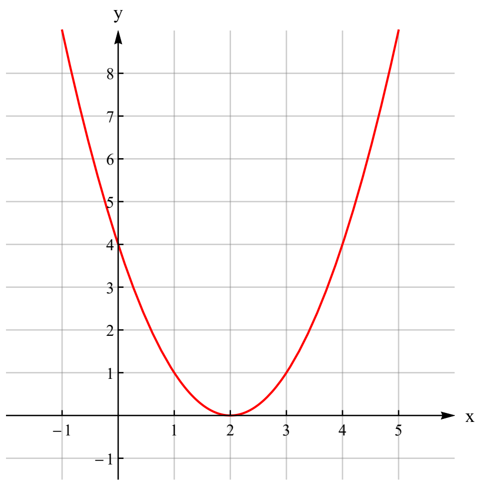
(b)



Graph of $g(x) = -2x^2 - 8x - 3$.

The y-intercept of this quadratic function is $y = -3$. The graph crosses the x-axis at two points so there are two x-intercepts. Visual inspection reveals that the approximate values of x at which the x-intercepts are located are $x = -3.6$ and $x = -0.4$. The vertex is the highest point on the graph since this is a downward facing parabola; hence the vertex is $(-2, 5)$.

(c)



Graph of $h(x) = x^2 - 4x + 4$.

The y-intercept of this function is $y = 4$. The graph crosses the x-axis at one point only: $(2, 0)$. Hence, $h(x)$ has one horizontal intercept $x = 2$. The vertex is the lowest point on the graph since this is an upward facing parabola; hence the vertex is $(2, 0)$.

The graph of a quadratic function $y = ax^2 + bx + c$ is a parabola. To be precise, we will call it a **quadratic parabola** to distinguish it from parabola-like graphs of other functions.

The examples above show that a quadratic function $f(x) = ax^2 + bx + c$ may have two horizontal intercepts, one horizontal intercept, or no horizontal intercepts. That means that a quadratic equation $ax^2 + bx + c = 0$ may have two solutions, one solution or no solutions at all.

Recall that horizontal intercepts of $f(x)$ are also called **zeros** of $f(x)$ as they are values of x where $f(x) = 0$. (Zeros of a polynomial function are often called **roots** of the function.)

The symmetry of a parabola about the vertical line passing through its vertex is well worth remembering.

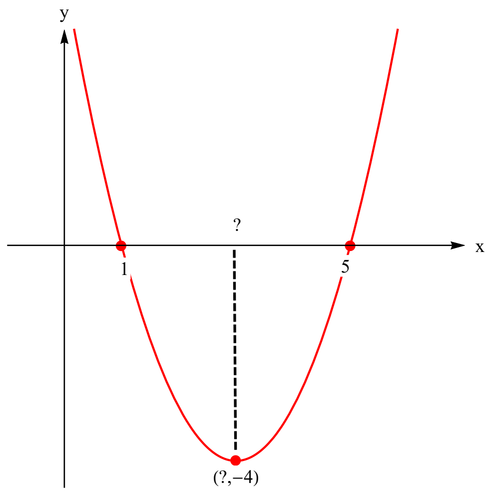
The Symmetry of a Parabola

A parabola $y = ax^2 + bx + c$ is symmetric about the vertical line passing through its vertex.

Hence, if the horizontal intercepts of a parabola $y = ax^2 + bx + c$ are $x = x_1$ and $x = x_2$, then the x-coordinate of the vertex is the midpoint between the horizontal intercepts:

$$\text{The } x\text{-coordinate of the vertex} = \text{The midpoint between the horizontal intercepts} = \frac{x_1 + x_2}{2}.$$

Example 3.1.2. Below you see the graph of a quadratic function $f(x) = ax^2 + bx + c$. The horizontal intercepts and the y-coordinate of the vertex are given. Find the x-coordinate of the vertex and the vertex itself.

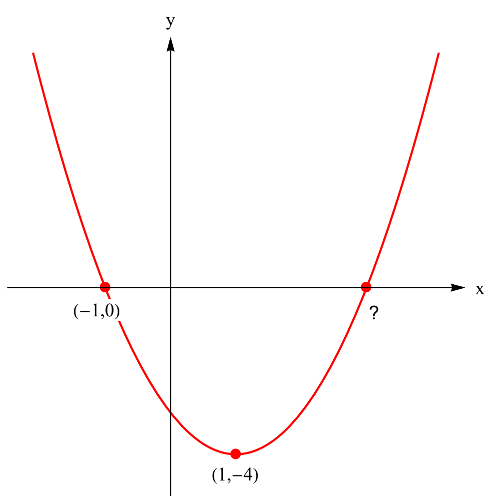


Solution. The x-coordinate of the vertex is the midpoint between $x = 1$ and $x = 5$. We can easily guess that the midpoint is $x = 3$. If it is hard to guess, use the formula for the midpoint:

$$\text{Midpoint} = \frac{1+5}{2} = 3.$$

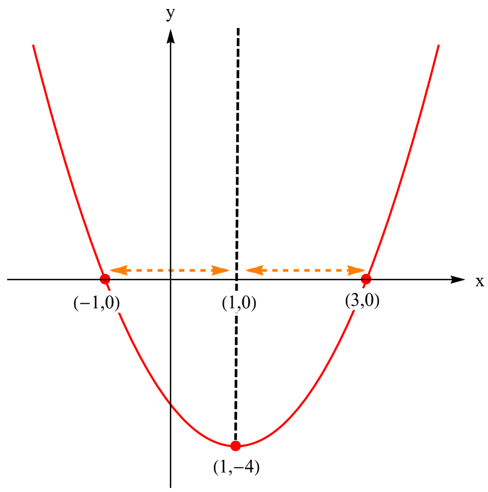
The x-coordinate of the vertex is 3. We read the y-coordinate of the vertex from the graph. The vertex is $(3, -4)$.

Example 3.1.3. Below you see the graph of a quadratic function $f(x) = ax^2 + bx + c$. The point on the x-axis, $(-1, 0)$, that corresponds to one of the two horizontal intercepts and the vertex of the parabola, $(1, -4)$, are marked. Find the point on the x-axis that corresponds to the second horizontal intercept of the function $f(x)$.



Solution. The graph of every quadratic function — every quadratic parabola — is symmetric with respect to the vertical line passing through its vertex. The vertical line through the vertex

$(1, -4)$ crosses the x-axis at the point $(1, 0)$. The point on the x-axis symmetric to the point $(-1, 0)$ about the vertical line is $(3, 0)$. Hence, the second horizontal intercept of the parabola is $x = 3$ and the corresponding point of the x-axis is $(3, 0)$:



Note that the x-coordinate of the vertex, $x = 1$, is the midpoint between the horizontal intercepts as it should be: $1 = \frac{-1+3}{2}$.

Quadratic functions often model real-life scenarios, as illustrated in the following example.

Example 3.1.4. A chair manufacturer finds that the number of chairs that it can sell depends on the price p (in dollars) that it charges per chair. Specifically, the number of chairs that will be sold if p dollars is charged per chair is given by the formula $1200 - 6p$.

(a) Find the formula for the revenue function $R(p)$ and graph $R(p)$.
(b) For what price(s) per chair the manufacturer's revenue is \$0?
(c) What is the maximum revenue? What price should the chair manufacturer charge per chair in order to maximize revenue?

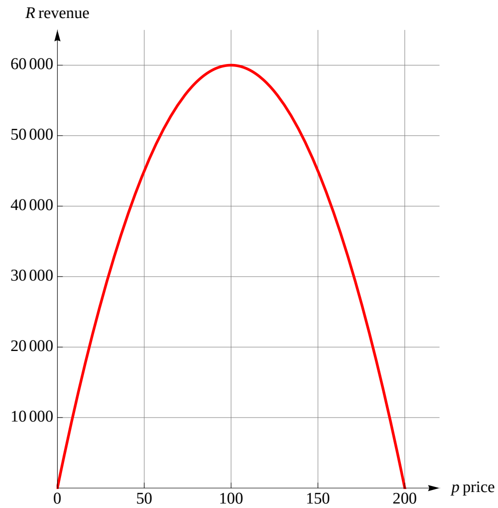
Solution. (a) The revenue $R(p)$ is the income that the chair manufacturer makes from selling chairs. This means that

$$\begin{aligned} R(p) &= \text{ (price charged per chair)} \times \text{ (number of chairs sold at that price point)} \\ &= p(1200 - 6p). \end{aligned}$$

Note that the revenue function is in fact a quadratic function, which can be seen more easily by using the distributive law to rewrite the formula we found above:

$$\begin{aligned} R(p) &= p(1200 - 6p) \\ &= 1200p - 6p^2 \\ &= -6p^2 + 1200p. \end{aligned}$$

The graph of the revenue function can be obtained using a table of values or a graphing calculator and is shown below.



Graph of $R(p) = p(1200 - 6p)$.

(b) The price(s) for which the chair manufacturer's revenue will be \$0 can be found by setting the revenue function equal to 0 and solving for p:

$$p(1200 - 6p) = 0.$$

The only way that a product of two factors can be 0 is if one of the factors itself is 0, so the above breaks into the two equations:

$$p = 0$$

$$\begin{aligned} 1200 - 6p &= 0 \\ 1200 - 1200 - 6p &= 0 - 1200 \\ \frac{-6p}{-6} &= \frac{-1200}{-6} \\ p &= 200. \end{aligned}$$

These two values can also be found by visually inspecting the graph of $R(p)$. The revenue is \$0 where the graph crosses the horizontal axis, which occurs at the values of $p = 0$ and $p = 200$.

Hence if either \$0 or \$200 is charged per chair, the revenue will be \$0. In the first case, if the chair manufacturer charges nothing for a product, it will of course receive no income. In the second case, charging \$200 per chair results in the chair manufacturer pricing itself out of the market; they have set the price too high for customers to be willing to purchase the item.

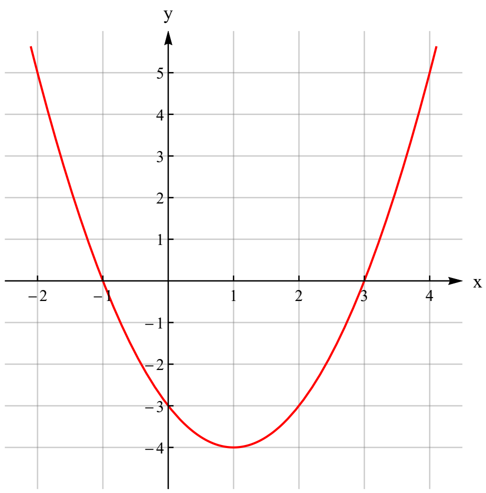
(c) Since the graph of $R(p)$ is a parabola that opens down, answering questions about the maximum revenue involves the vertex. Since the vertex is $(100, 60000)$, the maximum revenue occurs when a price of \$100 is charged per chair. The maximum revenue is \$60000.

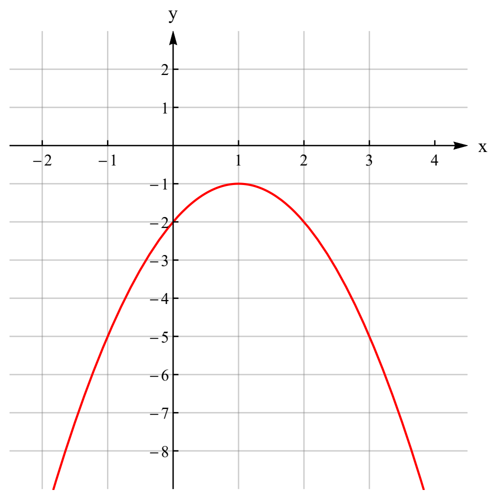
Practice Problems for Section 3.1

In Problems 1-4, write each quadratic function in standard form. Then determine whether its graph will open up or open down without graphing the function.

1. $f(x) = x(4x - 7)$
2. $h(t) = 17 + 3t - 5t^2 - 2(1.5t - 2)$
3. $g(x) = 4x(x - 4) - 2x(x - 1)$
4. $y = -4(x + 5)^2 + 80$

In Problems 5-6, use the graph of each quadratic function to identify its intercept(s) and vertex.

5. 

6. 

In Problems 7-8, the horizontal intercepts of a quadratic parabola are given. Find the x-coordinate of its vertex.

7. $x = -1.5$ and $x = 1.5$
8. $x = -2$ and $x = 5$

9. A quadratic parabola crosses the x-axis at the point $(2, 0)$ and has its vertex at the point $(4, 4)$. Find the second point on the x-axis where the parabola crosses the x-axis. Does the parabola open up or open down?

10. A lighting company can sell $1000 - 2p$ units of its specialty chandelier if it charges \$p per specialty chandelier.

    (a) Find the formula for the revenue function $R(p)$ and graph it.

    (b) At what price will the lighting company sell zero of its specialty chandelier?

    (c) What is the maximum revenue? What price should the lighting company charge per specialty chandelier in order to maximize revenue?

3.2 Factoring Quadratic Expressions

In order to algebraically solve for the horizontal intercepts of a quadratic function such as

$$f(x) = x^2 + 5x + 6,$$

one must set $f(x)$ equal to 0 and then solve for x; in other words, one must solve the quadratic equation

$$x^2 + 5x + 6 = 0.$$

This equation can be solved by **factoring**. This process will be explained below.

When we multiply together

$$(x + M)(x + N)$$

for some real numbers M and N, we obtain:

$$\begin{aligned}(x + M)(x + N) &= x^2 + Mx + Nx + M \cdot N \\ &= x^2 + (M + N)x + M \cdot N.\end{aligned}$$

So if one begins with a quadratic expression of the form $x^2 + Bx + C$ and wants to find two numbers M and N such that

$$x^2 + Bx + C = (x + M)(x + N),$$

it must be the case that $M + N = B$ and $M \cdot N = C$.

Factoring a Quadratic Expression of the Form $x^2 + Bx + C$

In order to factor a quadratic expression of the form x^2+Bx+C (where B, and C are constants and x is a variable), find two numbers M and N whose product is C and whose sum is B. Then $x^2 + Bx + C = (x + M)(x + N)$.

Example 3.2.1. Factor each quadratic expression.

(a) $x^2 + 5x + 6$ (b) $t^2 - 16t + 64$ (c) $w^2 + 8w - 20$

Solution. (a) We need to find two numbers whose product is $C = 6$ and whose sum is $B = 5$. Since $2 \cdot 3 = 6$ and $2 + 3 = 5$, we have

$$x^2 + 5x + 6 = (x + 2)(x + 3).$$

(b) We need to find two numbers whose product is $C = 64$ and whose sum is $B = -16$. Since $-8 \cdot (-8) = 64$ and $-8 + (-8) = -16$, we have

$$t^2 - 16t + 64 = (t - 8)(t - 8) = (t - 8)^2.$$

(c) We need to find two numbers whose product is $C = -20$ and whose sum is $B = 8$. Since $-2 \cdot 10 = -20$ and $-2 + 10 = 8$, we have

$$w^2 + 8w - 20 = (w - 2)(w + 10).$$

Example 3.2.2. Find the horizontal intercepts of each quadratic function.

(a) $f(x) = x^2 + 5x + 6$ (b) $y = z^2 - 49$

Solution. (a) To find the horizontal intercepts, we set the function equal to 0 and solve for x:

$$x^2 + 5x + 6 = 0.$$

We factored the left-hand side of this equation in Example 3.1.3(a), so

$$(x + 2)(x + 3) = 0$$

$$\begin{aligned} x + 2 &= 0 \\ x + 2 - 2 &= 0 - 2 \\ x &= -2 \end{aligned} \qquad\qquad \begin{aligned} x + 3 &= 0 \\ x + 3 - 3 &= 0 - 3 \\ x &= -3. \end{aligned}$$

Hence the horizontal intercepts of $f(x) = x^2 + 5x + 6$ are $x = -3$ and $x = -2$.

(b) Again, we set the quadratic function equal to 0 and solve:

$$z^2 - 49 = 0.$$

In order to factor the left-hand side, we note that $z^2 - 49 = z^2 + 0z - 49$, so we need two numbers whose product is $C = -49$ and whose sum is $B = 0$. Since $-7 \cdot 7 = -49$ and $-7 + 7 = 0$, we obtain

$$(z - 7)(z + 7) = 0$$

$$\begin{aligned} z - 7 &= 0 \\ z - 7 + 7 &= 0 + 7 \\ z &= 7 \end{aligned} \qquad\qquad \begin{aligned} z + 7 &= 0 \\ z + 7 - 7 &= 0 - 7 \\ z &= -7. \end{aligned}$$

Hence the horizontal intercepts of $y = z^2 - 49$ are $x = -7$ and $x = 7$.

The quadratic function $z^2 - 49$ in the above example is an example of a **difference of squares**.

Factoring a Difference of Squares

The difference of squares $A^2 - B^2$ factors into $(A + B)(A - B)$:

$$A^2 - B^2 = (A + B)(A - B).$$

The above formula can be verified quickly through multiplication:

$$\begin{aligned}(A + B)(A - B) &= A^2 - AB + AB - B^2 \\ &= A^2 - B^2.\end{aligned}$$

Example 3.2.3. Factor each difference of squares:

(a) $16x^2 - 81$ (b) $100y^2 - \frac{1}{4}$

Solution. (a) Here,

$$16x^2 - 81 = (4x)^2 - (9)^2$$

so

$$16x^2 - 81 = (4x + 9)(4x - 9).$$

(b) Here,

$$100y^2 - \frac{1}{4} = (10y)^2 - \left(\frac{1}{2}\right)^2$$

so

$$100y^2 - \frac{1}{4} = \left(10y + \frac{1}{2}\right)\left(10y - \frac{1}{2}\right).$$

Each of the above factoring strategies are special cases of a more general approach to factoring quadratic functions.

> **Factoring a Quadratic Expression of the Form $ax^2 + bx + c$**
>
> To factor a trinomial of the form $ax^2 + bx + c$ (where a, b, and c are constants with $a \neq 0$ and x is a variable), **rewrite the term** bx to factor by grouping. This is done by finding two numbers **whose product is** $a \cdot c$ **and whose sum is** b. These two numbers can be used to rewrite the term bx.

Example 3.2.4. Factor each of the following quadratic expressions.

(a) $x^2 + 5x - 36$ (b) $6x^2 + 7x - 20$

Solution. (a) Here $a = 1$, $b = 5$, and $c = -36$. List all **factors of** $a \cdot c = -36$. Notice that the pair $9, -4$ **multiply to** -36 and **sum to** 5. Now

$$\begin{aligned}x^2 + 5x - 36 &= x^2 + 9x - 4x - 36 \\ &= x(x + 9) - 4(x + 9) \\ &= (x + 9)(x - 4).\end{aligned}$$

(b) Here $a = 6$, $b = 7$, and $c = -20$. List all **factors of** $a \cdot c = -120$. Notice that the pair $15, -8$ **multiply to** -120 and **sum to** 7. Now

$$\begin{aligned}6x^2+7x-20 &= 6x^2+15x-8x-20\\ &= 3x(2x+5)-4(2x+5)\\ &= (2x+5)(3x-4).\end{aligned}$$

Example 3.2.5. Find the horizontal intercepts of the quadratic function $y=-3t^2-t+2$.

Solution. We set the quadratic function equal to 0 and solve:

$$-3t^2-t+2=0.$$

In order to factor the left-hand side, we observe that $a=-3$, $b=-1$, and $c=2$. List all **factors of** $a\cdot c=-6$. Notice that the pair $-3, 2$ **multiply to** -6 and **sum to** -1. Now

$$\begin{aligned}-3t^2-t+2 &= -3t^2-3t+2t+2\\ &= -3t(t+1)+2(t+1)\\ &= (t+1)(-3t+2).\end{aligned}$$

so we have that

$$(t+1)(-3t+2)=0$$

$$\begin{aligned}t+1&=0\\ t+1-1&=0-1\\ t&=-1\end{aligned}$$

$$\begin{aligned}-3t+2&=0\\ -3t+2-2&=0-2\\ -3t&=-2\\ \frac{\cancel{-3}t}{\cancel{-3}}&=\frac{-2}{-3}.\\ t&=\frac{2}{3}\end{aligned}$$

Hence, the horizontal intercepts of $y=-3t^2-t+2$ are $t=-1$ and $t=2/3$. The function $y=-3t^2-t+2$ can be factored in the form $y=(t+1)(-3t+2)$. By factoring out -3 from the second term, we can factor the function even further as $y=-3(t+1)(t-\frac{2}{3})$. The latter form shows clearly the horizontal intercepts of the function; that is, the zeros of the function.

It is worth to mention another special case of factoring a quadratic expression ax^2+bx+c: the case when c is zero.

Factoring a Quadratic Expression of the Form ax^2+bx

Consider an expression of the form ax^2+bx (where a and b are constants with $a\neq 0$ and x is a variable). To factor the expression, factor out x from both terms:

$$ax^2+bx=x(ax+b)$$

or, alternatively, factor out ax from both terms:

$$ax^2+bx=ax(x+\frac{b}{a}).$$

If we rewrite a given expression $ax^2 + bx$ as $ax(x + \frac{b}{a})$, the horizontal intercepts of the function $y = ax^2 + bx$ are very easy to find. We want to solve the equation:

$$ax(x + \frac{b}{a}) = 0.$$

The product is equal to 0 if $ax = 0$ or $x + \frac{b}{a} = 0$. Since $a \neq 0$, $ax = 0$ when $x = 0$. The second term $x + \frac{b}{a} = 0$ when $x = -\frac{b}{a}$. Hence, we have two x-intercepts for $ax^2 + bx$:

$$x = 0 \qquad \text{and} \qquad x = -\frac{b}{a}.$$

Example 3.2.6. Find horizontal intercepts of each of the quadratic functions given below.

(a) $y = -2x^2 + 3x$ (b) $y = x^2 - 5x$

Solution. (a) $a = -2$, $b = 3$, and $c = 0$. We factor $-2x$ from both terms and obtain:

$$-2x^2 + 3x = -2x(x + (-\frac{3}{2})) = -2x(x - \frac{3}{2}).$$

The solutions to the equation:

$$-2x(x - \frac{3}{2}) = 0$$

are $x = 0$ and $x = \frac{3}{2}$ and these are the two horizontal intercepts of the function.

(b) $a = 1$, $b = -5$, and $c = 0$. We factor out x from both terms and obtain: $x^2 - 5x = x(x - 5)$. The horizontal intercepts are $x = 0$ and $x = -\frac{b}{a} = 5$.

Consider a projectile in a vertical motion up or down. The height of the projectile above the ground, $H(t)$, in feet, t seconds after it has been thrown can be modeled by the quadratic function

$$H(t) = -16t^2 + V_0 t + H_0$$

where V_0 is the initial velocity of the object in feet per second and H_0 is the initial height of the object in feet.

Example 3.2.7. A model rocket is launched from the ground with an initial velocity of 180 feet per second. Let $H(t)$ be the height of the rocket above the ground t seconds after launch.

(a) Write a formula for the function $H(t)$.

(b) Find t when the model rocket hits the ground.

(c) Find the maximum height that the model rocket reaches.

Solution. (a) The model rocket is launched from the ground. Hence, the initial height H_0 is 0 feet: $H_0 = 0$. The initial velocity is 180 ft/sec. Hence, $V_0 = 180$. The function that gives the height of the rocket after t seconds is:

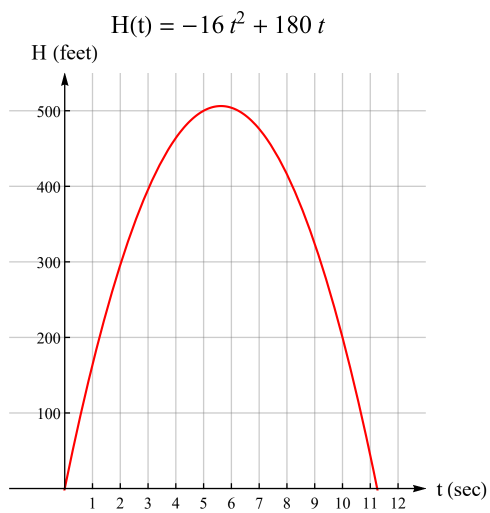
$$H(t) = -16t^2 + 180t.$$

(b) The rocket is traveling vertically straight up, reaches its maximum height, and then falls back to the ground. It hits the ground at the time t when its height above the ground is zero; that is, when $H(t) = 0$. To find such t, we factor the height function and set it equal to 0 to find its horizontal intercepts.

$$H(t) = -16t^2 + 180t = t(-16t + 180) = -16t(t + \frac{180}{-16}) = -16t(t - 11.25).$$

From the factored form, we see the horizontal intercepts. The height $H(t) = -16t(t - 11.25) = 0$ at $t = 0$ — at the moment of launch — and at $t = 11.25$ when the rocket comes back and hits the ground. Hence, the rockets hits the ground 11.25 seconds after launch.

(c) The rocket reaches its maximum height when $H(t)$ is at its maximum. The graph of $H(t) = -16t^2 + 180t$ is a parabola opening down. Hence, $H(t)$ is at its maximum at the vertex of the parabola. Remember that the t-coordinate of the vertex is in the middle between the horizontal intercepts, $t = 0$ and $t = 11.25$. Hence, the t-coordinate of the vertex is $\frac{0+11.25}{2} = 5.75$. Thus, the maximum height is reached at time $t = 5.75$. The height at $t = 5.75$ is $H(5.75) = -16 \cdot 5.75^2 + 180 \cdot 5.75 = 506$ feet. So at its highest, the rocket reaches 506 feet. The graph of the function $H(t)$ explains everything:



Note that the graph shows the height of the rocket $H(t)$ as a function of time t. It does not show the path of the rocket. The rocket travels up and down along a vertical line.

Practice Problems for Section 3.2

In Problems 1-12 find the intercepts of each quadratic function.

1. $f(x) = x^2 + 3x + 2$
2. $y = x^2 - x - 6$
3. $g(t) = t^2 + t - 6$
4. $y = x^2 - 8x + 16$
5. $f(w) = w^2 + 14w + 49$
6. $f(t) = 64t^2 - 36$
7. $h(x) = 2x^2 + 10x + 8$
8. $y = 2w^2 - w - 3$
9. $f(x) = -x^2 - 6x + 16$
10. $g(x) = \frac{1}{49} - 4x^2$
11. $p(x) = 2x^2 - 6x$
12. $m(t) = -5t^2 + 2t$
13. A model rocket is launched vertically from the ground with an initial velocity of 168 feet per second. Let $H(t)$ be the height, in feet, of the rocket t seconds after launch.
    (a) Write a formula for the function $H(t)$.
    (b) Find t when the model rocket hits the ground.
    (c) Find the maximum height that the model rocket reaches.
    Use the graph of $H(t)$ to help you solve the problem.

3.3 Vertex Form of a Quadratic Function, Completing the Square

Recall that the standard form is a quadratic function is $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers with $a \neq 0$.

When one first looks upon the function

$$f(x) = (x - 1)^2 + 2$$

it may not immediately be apparent that this is in fact a quadratic function. This can be seen by expanding the right-hand side:

$$\begin{aligned} f(x) &= (x-1)^2 + 2 \\ &= (x-1)(x-1) + 2 \\ &= x^2 - x - x + 1 + 2 \\ &= x^2 - 2x + 3. \end{aligned}$$

Hence $f(x) = (x - 1)^2 + 2$ can be written in the standard form of a quadratic function as

$$f(x) = x^2 - 2x + 3.$$

This is the same quadratic function that appeared in Example 3.1.1 where we saw that the vertex was $(1, 2)$. Notice that these numbers appear in the alternate formula for $f(x) = x^2 - 2x + 3$:

$$f(x) = (x - 1)^2 + 2.$$

This is not a coincidence and brings us to the **vertex form** of a quadratic function.

Vertex Form of a Quadratic Function

The **vertex form** of a quadratic function with vertex (h, k) is given by

$$f(x) = a(x - h)^2 + k.$$

Example 3.3.1. Identify the vertex of each of the following quadratic functions.

(a) $f(t) = -3(t-6)^2 - 4$ (b) $y = (x+7)^2 + 3$

Solution. (a) The function can be written as

$$f(t) = -3(t - 6)^2 + (-4).$$

It can be seen that $h = 6$ and $k = -4$, so its vertex is $(6, -4)$.

(b) Note that for the function

$$y = (x + 7)^2 + 3,$$

addition is being performed inside of the parentheses instead of subtraction, as is required to correctly identify h using the vertex form. Recalling that subtracting a negative number is the same as adding, we can rewrite the above as

$$y = (x - (-7))^2 + 3$$

which allows us to identify that $h = -7$ and $k = 3$. The vertex is $(-7, 3)$.

Converting from Vertex Form to Standard Form

In the beginning of this section, we converted $f(x) = (x - 1)^2 + 2$ from vertex form to its standard form $f(x) = x^2 - 2x + 3$ by expanding the right-hand side. This is what is done in general to convert a quadratic function from vertex form to standard form. To expedite this process, special product formulas may be used. Note that

$$\begin{aligned}(A+B)^2 &= (A+B)(A+B)\\ &= A^2 + AB + AB + B^2\\ &= A^2 + 2AB + B^2\end{aligned} \quad \text{while} \quad \begin{aligned}(A-B)^2 &= (A-B)(A-B)\\ &= A^2 - AB - AB + B^2\\ &= A^2 - 2AB + B^2.\end{aligned}$$

Special Product Formulas

- $(A + B)^2 = A^2 + 2AB + B^2$
- $(A - B)^2 = A^2 - 2AB + B^2$

Example 3.3.2. Convert each quadratic function from vertex form to standard form and identify a, b, and c.

(a) $y = \left(x - \frac{1}{2}\right)^2 + 5$ (b) $g(t) = -3(t-5)^2 - 10$

Solution. (a) Note that

$$\begin{aligned} y &= \left(x + \frac{1}{2}\right)^2 + 5 \\ &= \left((x)^2 + 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2\right) + 5 \\ &= x^2 + x + \frac{1}{4} + 5 \\ &= x^2 + x + \frac{1}{4} + \frac{5}{1} \cdot \frac{4}{4} \\ &= x^2 + x + \frac{1}{4} + \frac{20}{4} \\ &= x^2 + x + \frac{21}{4} \end{aligned}$$

so the standard form is $y = x^2 + x + \frac{21}{4}$. Here $a = 1$, $b = 1$, and $c = \frac{21}{4}$.

(b) Note that

$$\begin{aligned} g(t) &= -3(t-5)^2 - 10 \\ &= -3\left((t)^2 - 2(t)(5) + (5)^2\right) - 10 \\ &= -3(t^2 - 10t + 25) - 10 \\ &= -3(t^2 - 10t + 25) - 10 \\ &= -3t^2 + 30t - 75 - 10 \\ &= -3t^2 + 30t - 85 \end{aligned}$$

so the standard form is $g(t) = -3t^2 + 30t - 85$. Here $a = -3$, $b = 30$, and $c = -85$.

Converting from Standard Form to Vertex Form

Converting from standard form to vertex form requires **completing the square**. This process will be explained below, but first an observation. Note that

$$\begin{aligned} x^2 + Bx + \left(\frac{B}{2}\right)^2 &= x^2 + 2(x)\left(\frac{B}{2}\right) + \left(\frac{B}{2}\right)^2 \\ &= \left(x + \frac{B}{2}\right)^2 \end{aligned}$$

by the first special product formula.

Completing the Square on a Quadratic of the Form $x^2 + Bx + C$

To complete the square on a quadratic function of the form $x^2 + Bx + C$:

1. Identify B, the coefficient of the "x" term.
2. Compute $\left(\frac{B}{2}\right)^2$.
3. Add and subtract this quantity from the original quadratic function:
$$x^2 + Bx + \left(\frac{B}{2}\right)^2 + C - \left(\frac{B}{2}\right)^2.$$
Note that by both adding and subtracting the same number from the original function, you have in fact added 0, which does not change the function.
4. Factor $x^2 + Bx + \left(\frac{B}{2}\right)^2$ into $\left(x + \frac{B}{2}\right)^2$ and combine constants to finish completing the square.

Example 3.3.3. Convert each of the following quadratic functions from standard form to vertex form. Then identify the vertex.

(a) $f(x) = x^2 + 6x + 10$ (b) $y = t^2 - 8t + 1$ (c) $g(x) = x^2 + 5x - 3$

Solution. (a) The quadratic function $f(x) = x^2 + 6x + 10$ is of the form $x^2 + Bx + C$ with $B = 6$. Now
$$\left(\frac{B}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = (3)^2 = 9.$$
Adding and subtracting this quantity from the quadratic yields:
$$\begin{aligned} x^2 + 6x + 10 &= \underbrace{x^2 + 6x + 9}_{\text{factors into } (x+3)^2} + 10 - 9 \\ &= (x+3)^2 + 1. \end{aligned}$$
Hence the vertex form of $f(x) = x^2 + 6x + 10$ is
$$f(x) = (x+3)^2 + 1$$
and the vertex is $(-3, 1)$.

(b) The quadratic function $y = t^2 - 8t + 1$ is of the form $t^2 + Bt + C$ with $B = -8$. Now
$$\left(\frac{B}{2}\right)^2 = \left(\frac{-8}{2}\right)^2 = (-4)^2 = 16.$$
Adding and subtracting this quantity from the quadratic yields:
$$\begin{aligned} t^2 - 8t + 1 &= \underbrace{t^2 - 8t + 16}_{\text{factors into } (t-4)^2} + 1 - 16 \\ &= (t-4)^2 - 15. \end{aligned}$$

Hence the vertex form of $y = t^2 - 8t + 1$ is

$$y = (t-4)^2 - 15$$

and the vertex is $(4, -15)$.

(c) The quadratic function $g(x) = x^2 + 5x - 3$ is of the form $x^2 + Bx + C$ with $B = 5$. Now

$$\left(\frac{B}{2}\right)^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$$

Adding and subtracting this quantity from the quadratic yields:

$$\begin{aligned}
x^2 + 5x - 3 &= \underbrace{x^2 + 5x + \frac{25}{4}}_{\text{factors into } (x+\frac{5}{2})^2} - 3 - \frac{25}{4} \\
&= \left(x + \frac{5}{2}\right)^2 - \frac{3}{1}\cdot\frac{4}{4} - \frac{25}{4}. \\
&= \left(x + \frac{5}{2}\right)^2 - \frac{12}{4} - \frac{25}{4}. \\
&= \left(x + \frac{5}{2}\right)^2 - \frac{37}{4}.
\end{aligned}$$

Hence the vertex form of $g(x) = x^2 + 5x - 3$ is

$$g(x) = \left(x + \frac{5}{2}\right)^2 - \frac{37}{4}$$

and the vertex is $\left(\frac{5}{2}, -\frac{37}{4}\right)$.

Completing the Square on a Quadratic of the Form $ax^2 + bx + c$ where $a \neq 1$

To complete the square on a quadratic function of the form $ax^2 + bx + c$ where $a \neq 1$:

1. Factor a out of the first two terms to obtain an expression of the form

$$a\left(x^2 + \frac{b}{a}x\right) + c$$

2. Complete the square within the parentheses.
3. Distribute a and combine constants to finish completing the square on the original quadratic.

Example 3.3.4. Convert each of the following quadratic functions from standard form to vertex form. Then identify the vertex.

(a) $y = 3x^2 - 6x + 1$ (b) $f(t) = -2t^2 - t - 4$

Solution. (a) We begin by factoring the 3 out of the first two terms to obtain

$$3(x^2 - 2x) + 1.$$

We then complete the square inside the parentheses; in other words we will complete the square on $x^2 - 2x$. Here $B = -2$ so

$$\left(\frac{B}{2}\right)^2 = \left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$$

and

$$\begin{aligned} x^2 - 2x &= \underbrace{x^2 - 2x + 1}_{\text{factors into } (x-1)^2} \quad - 1 \\ &= (x-1)^2 - 1. \end{aligned}$$

Now

$$\begin{aligned} 3(x^2 - 2x) + 1 &= 3((x-1)^2 - 1) + 1 \\ &= 3((x-1)^2 - 1) + 1 \\ &= 3(x-1)^2 - 3 + 1 \\ &= 3(x-1)^2 - 2. \end{aligned}$$

Hence the vertex form of $y = 3x^2 - 6x + 1$ is

$$y = 3(x-1)^2 - 2$$

and the vertex is $(1, -2)$.

(b) We begin by factoring the -2 out of the first two terms to obtain

$$-2\left(t^2 + \frac{1}{2}t\right) - 4.$$

We then complete the square inside the parentheses; in other words we will complete the square on $t^2 + \frac{1}{2}t$. Here $B = \frac{1}{2}$ so

$$\left(\frac{B}{2}\right)^2 = \left(\frac{1/2}{2}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

and

$$\begin{aligned} t^2 + \frac{1}{2}t &= \underbrace{t^2 + \frac{1}{2}t + \frac{1}{16}}_{\text{factors into } (t+\frac{1}{4})^2} \quad - \frac{1}{16} \\ &= \left(t + \frac{1}{4}\right)^2 - \frac{1}{16}. \end{aligned}$$

Now

$$
\begin{aligned}
-2\left(t^2+\frac{1}{2}t\right)-4 &= -2\left(\left(t+\frac{1}{4}\right)^2-\frac{1}{16}\right)-4 \\
&= -2\left(\left(t+\frac{1}{4}\right)^2-\frac{1}{16}\right)-4 \\
&= -2\left(x+\frac{1}{2}\right)^2+\frac{2}{16}-4 \\
&= -2\left(x+\frac{1}{2}\right)^2+\frac{1}{8}-\frac{4}{1}\cdot\frac{8}{8} \\
&= -2\left(x+\frac{1}{2}\right)^2+\frac{1}{8}-\frac{32}{8} \\
&= -2\left(x+\frac{1}{2}\right)^2-\frac{31}{8}.
\end{aligned}
$$

Hence the vertex form of $f(t) = -2t^2 - t - 4$ is

$$f(t) = -2\left(x+\frac{1}{2}\right)^2-\frac{31}{8}$$

and the vertex is $\left(-\frac{1}{2}, -\frac{31}{8}\right)$.

Using Vertex Form to Find Horizontal Intercepts

If a quadratic function is given in vertex form, then its horizontal intercepts can be found by following the process illustrated in example below.

Example 3.3.5. Find the horizontal intercepts of each of the following functions.

(a) $f(x) = x^2 - 9$ (b) $g(x) = (x+1)^2 - 10$ (c) $y(t) = 4(t-1)^2 - 64$
(d) $h(x) = (x-5)^2 + 8$

Solution. (a) To find the horizontal or x-intercept(s) of this quadratic function, we set $f(x)$ equal to 0, $f(x) = 0$, and solve for x. We could treat the left-hand side as a difference of squares and proceed by factoring, but we also could take an alternate approach:

$$
\begin{aligned}
x^2 - 9 &= 0 \\
x^2 - 9 + 9 &= 0 + 9 \\
x^2 &= 9.
\end{aligned}
$$

This equation can be solved by taking the square root of both sides, remembering that there will be both a positive solutions and a negative solution since both $(\sqrt{9})^2 = 9$ and $(-\sqrt{9})^2 = 9$:

$$
\begin{aligned}
x^2 &= 9 \\
x &= \pm\sqrt{9} \\
x &= \pm 3.
\end{aligned}
$$

Hence the horizontal intercepts of the function $f(x) = x^2 - 9$ and the solutions of the equation $x^2 - 9 = 0$ are $x = -3$ and $x = 3$.

(b) To find the horizontal intercepts, we set $g(x)$ equal to 0 and solve for x:

$$\begin{aligned}
(x+1)^2 - 10 &= 0 \\
(x+1)^2 - 10 + 10 &= 0 + 10 \\
(x+1)^2 &= 10 \\
x + 1 &= \pm\sqrt{10} \\
x + 1 - 1 &= \pm\sqrt{10} - 1 \\
x &= -1 \pm \sqrt{10}.
\end{aligned}$$

We obtain two solutions to our equation and two horizontal intercepts of $g(x)$: $x = -1 - \sqrt{10}$ and $-1 + \sqrt{10}$.

(c) To find the horizontal intercepts, we set $y(t)$ equal to 0 and solve for t:

$$\begin{aligned}
4(t-1)^2 - 3 + 64 &= 0 + 64 \\
\frac{\cancel{4}(t-1)^2}{\cancel{4}} &= \frac{64}{4} \\
(t-1)^2 &= 16 \\
t - 1 &= \pm\sqrt{16} \\
t - 1 &= \pm 4 \\
t - 1 + 1 &= \pm 4 + 1 \\
t &= 1 \pm 4.
\end{aligned}$$

Hence, the two solutions and two horizontal intercepts are $t = 1 + 4 = 5$ and $t = 1 - 4 = -3$.

(d) Once more, we set the function $h(x)$ equal to 0 and solve for x:

$$\begin{aligned}
(x-5)^2 + 8 &= 0 \\
(x-5)^2 + 8 - 8 &= 0 - 8 \\
(x-5)^2 &= -8.
\end{aligned}$$

In this case there is no real solution, since $\sqrt{-8}$ is not a real number. This means that the equation has no (real) solutions and the function $h(x) = (x-5)^2 + 8$ has no horizontal intercepts. If we take a moment to visualize the graph of the function, this makes sense. Its graph is an upward facing parabola with vertex $(5, 8)$. Since the lowest point (the vertex) of this parabola is above the x-axis, there cannot be any horizontal intercepts.

Practice Problems for Section 3.3

In Problems 1-4, identify the vertex of the given quadratic function.

1. $f(x) = -2(x-3)^2 + 4$

2. $y = 5(x+2)^2 - 3$

3. $g(t) = 2(t-1)^2 - 7$

4. $h(x) = -(x+\frac{1}{2})^2 + \frac{5}{3}$

In Problems 5-8, convert each quadratic function from vertex form to standard form and identify a, b, and c.

5. $f(x) = (x+1)^2 + 2$

6. $g(x) = 5(x-2)^2 + 6$

7. $y = -2(t-4)^2 + 5$

8. $h(x) = -(x+\frac{1}{2})^2 + \frac{5}{3}$

In Problems 9-14, convert each quadratic function from standard form to vertex form by completing the square. Then identify the vertex.

9. $f(x) = x^2 + 2x - 5$

10. $g(x) = x^2 - 6x + 10$

11. $h(x) = x^2 + 3x + \frac{5}{4}$

12. $F(x) = -x^2 + 10x + 15$

13. $G(x) = -3x^2 + 6x - 4$

14. $H(x) = -2x^2 - 16x - 17$

In Problems 15-18, find the horizontal intercepts of each quadratic function. Give exact and approximate values rounded off to three decimal places.

15. $f(x) = (x+3)^2 - 16$

16. $g(x) = (x-1)^2 - 2$

17. $h(x) = (x-\frac{1}{2})^2 - 1$

18. $F(x) = x^2 + 4x - 6$

19. $G(x) = 2x^2 - 3x + 4$

20. $H(x) = -2x^2 - 16x - 27$

3.4 The Quadratic Formula

The Quadratic Formula

In the previous section, we saw that finding the horizontal intercepts of a quadratic function in the vertex form $f(x) = a(x-h)^2 + k$ requires setting the quadratic function equal to 0, $f(x) = 0$, and solving. If we wanted to find the horizontal intercepts of a quadratic function given in the standard form $f(x) = ax^2 + bx + c$, we could proceed by first completing the square to convert to vertex form, and then proceed as before. If this process is done in general; i.e. on $f(x) = ax^2 + bx + c$ (without replacing a, b, and c with numbers), we obtain the **quadratic formula**.

The Quadratic Formula

A **quadratic equation in standard form** is written in the form

$$ax^2 + bx + c = 0.$$

The solution(s) to a quadratic equation in standard form are given by the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 3.4.1. Use the quadratic formula to solve each quadratic equation.

(a) $x^2 + x - 6 = 0$ (b) $-2t^2 - 2t + 7 = 0$ (c) $(x + 1)(x - 2) = 4$

Solution. (a) Since this quadratic equation is in standard form, $a = 3$, $b = -5$, and $c = 4$. Now by the quadratic formula

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-6)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{25}}{2} \\ &= \frac{-1 \pm 5}{2}. \end{aligned}$$

So the two solutions are $x = \dfrac{-1 - 5}{2} = \dfrac{-6}{2} = -3$ and $x = \dfrac{-1 + 5}{2} = \dfrac{4}{2} = 2$.

(b) Since this quadratic equation is in standard form, $a = -2$, $b = -2$, and $c = 7$. Now by the quadratic formula

$$\begin{aligned} t &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(-2)(7)}}{2(-2)} \\ &= \frac{2 \pm \sqrt{60}}{-4} \\ &= \frac{2 \pm \sqrt{4 \cdot 15}}{-4}. \\ &= \frac{2 \pm 2\sqrt{15}}{-4} \end{aligned}$$

So the two solutions are

$$t = \frac{2 - 2\sqrt{15}}{-4} = \frac{2}{-4} + \frac{2\sqrt{15}}{4} = -\frac{1}{2} + \frac{\sqrt{15}}{2} \approx 1.4365$$

and

$$t = \frac{2 + 2\sqrt{15}}{-4} = \frac{2}{-4} - \frac{2\sqrt{15}}{4} = -\frac{1}{2} - \frac{\sqrt{15}}{2} \approx -2.4365.$$

(c) Since this quadratic equation is **not** in standard form, we must first expand the left-hand side and then get all terms on one side before we are able to proceed:

$$
\begin{aligned}
(x-1)(x+2) &= 4 \\
x^2 + 2x - x - 2 &= 4 \\
x^2 + x - 2 &= 4 \\
x^2 + x - 2 - 4 &= 4 - 4. \\
x^2 + x - 6 &= 0
\end{aligned}
$$

Now that the equation is in standard form, we can see that this is exactly the same equation as in part (a), so the solutions are $x = -3$ and $x = 2$.

Example 3.4.2. Find the horizontal intercepts of the quadratic function $g(x) = -2x^2 - 8x - 3$.

Solution. To find the horizontal intercepts, we set the quadratic function equal to 0 and solve for x:

$$-2x^2 - 8x - 3 = 0.$$

Here $a = -2$, $b = -8$, and $c = -3$, so by the quadratic formula,

$$
\begin{aligned}
x &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(-2)(-3)}}{2(-2)} \\
&= \frac{8 \pm \sqrt{40}}{-4} \\
&= \frac{8 \pm \sqrt{4 \cdot 10}}{-4} \\
&= \frac{8 \pm 2\sqrt{10}}{-4}.
\end{aligned}
$$

Simplifying, the two solutions are

$$x = \frac{8 - 2\sqrt{10}}{-4} = \frac{8}{-4} + \frac{2\sqrt{10}}{4} = -2 + \frac{\sqrt{10}}{2} \approx -0.4189$$

and

$$x = \frac{8 + 2\sqrt{10}}{-4} = \frac{8}{-4} - \frac{2\sqrt{10}}{4} = -2 - \frac{\sqrt{10}}{2} \approx -3.5811.$$

Hence the horizontal intercepts are $-2 + \dfrac{\sqrt{10}}{2}$ and $-2 - \dfrac{\sqrt{10}}{2}$.

The Discriminant

How many solutions a quadratic equation in standard form will have is determined entirely by the quantity under the radical in the quadratic formula; we call this quantity the **discriminant**.

The Discriminant

For the quadratic equation in standard form $ax^2+bx+c=0$, the **discriminant** D is given by

$$D = b^2 - 4ac.$$

- If $D > 0$, then the quadratic equation has two distinct real solutions.
- If $D = 0$, then the quadratic equation has one real solution.
- If $D < 0$, then the quadratic equation has no real solutions.

Example 3.4.3. Use the discriminant to determine how many solutions each quadratic equation has.

(a) $x^2 - 4x + 4 = 0$ (b) $-3t^2 + t - 10 = 0$ (c) $-2t^2 - 2t + 7 = 0$

Solution. (a) Here, $a = 1$, $b = -4$, and $c = 4$, so $D = (-4)^2 - 4(1)(4) = 0$. Hence this quadratic equation has **one real solution**. By the quadratic formula, that solution is

$$x = \frac{-(-4) \pm \sqrt{0}}{2(1)} = \frac{4}{2} = 2.$$

(b) Here, $a = -3$, $b = 1$, and $c = -10$, so $D = (1)^2 - 4(-3)(-10) = -119$. Since $D < 0$, this quadratic equation has **no real solutions**. This can be seen by using the quadratic formula as well, yielding

$$x = \frac{-1 \pm \sqrt{-119}}{2(-3)}$$

which is not a real quantity due to the negative number underneath the square root.

(c) Here, $a = -2$, $b = -2$, and $c = 7$, so $D = (-2)^2 - 4(-2)(7) = 60$. Since $D > 0$, this quadratic equation has **two distinct real solutions**. We found these solutions in Exercise 3.3.1(b); they are $t = -\frac{1}{2} - \frac{\sqrt{15}}{2}$ and $t = -\frac{1}{2} + \frac{\sqrt{15}}{2}$.

Using Horizontal Intercepts to Find the Vertex

In Section 3.1 we noticed that quadratic functions are symmetric about the vertical line passing through the vertex. It is the case therefore that, in terms of its horizontal position, the vertex must lie exactly halfway between the horizontal intercepts. Hence, given a quadratic function $f(x) = ax^2 + bx + c$, the x-coordinate of the vertex must be the average of the function's horizontal intercepts, in other words, of the function's zeros. As before let's denote the x-coordinate of the vertex by h. Since the zeros are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$ per the quadratic formula, the x-coordinate of the vertex is given by:

$$\begin{aligned}
h &= \left(\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}\right) \Big/ 2 \\
&= \left(\frac{-b}{2a} + \cancel{\frac{\sqrt{b^2-4ac}}{2a}} + \frac{-b}{2a} - \cancel{\frac{\sqrt{b^2-4ac}}{2a}}\right) \Big/ 2 \\
&= \left(\frac{-\cancel{2}b}{\cancel{2}a}\right) \Big/ 2 \\
&= -\frac{b}{a} \cdot \frac{1}{2} \\
&= -\frac{b}{2a}.
\end{aligned}$$

The above formula for h remains valid even if a quadratic function $f(x) = ax^2 + bx + c$ does not have horizontal intercepts or zeros.

Obtaining the Vertex from Standard Form

The vertex (h, k) of a quadratic function in the standard form $f(x) = ax^2 + bx + c$ satisfies that

$$h = -\frac{b}{2a} \qquad \text{and} \qquad k = f(h) = f\left(-\frac{b}{2a}\right).$$

Example 3.4.4. A baseball is thrown straight up with the initial speed of 40 ft/sec by a player who is 6 feet tall. Let $H(t)$ be the height of the ball above the ground t seconds later.

(a) Find a formula for $H(t)$.

(b) When does the ball reach its maximum height and what is its maximum height?

Solution. (a) We use the formula for the vertical motion of a projectile with initial velocity V_0 and initial height H_0:

$$H(t) = -16t^2 + V_0 t + H_0.$$

In our example, $H_0 = 6$ and $V_0 = 40$. Hence:

$$H(t) = -16t^2 + 40t + 6.$$

$H(t)$ is a quadratic function with $a = -16$, $b = 40$, and $c = 6$.

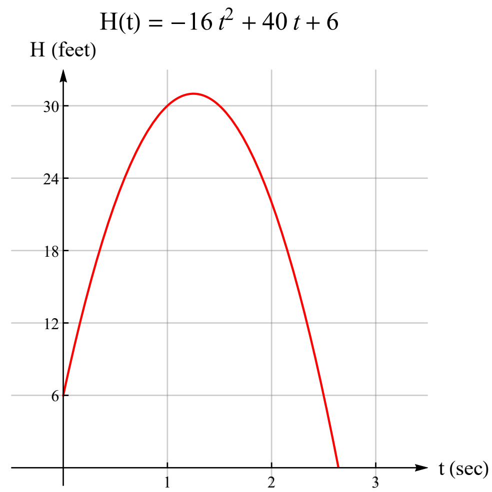
(b) The parabola corresponding to the function $H(t)$ opens down so $H(t)$ is at its maximum at the vertex of the parabola. Given $a = -16$, $b = 40$, we can easily calculate h; that is, the t-coordinate of the vertex:

$$\text{The } t\text{-coordinate of the vertex } = h = -\frac{b}{2a} = -\frac{40}{2 \cdot (-16)} = 1.25.$$

The ball will reach its maximum height at $t = 1.25$; that is, 1.25 seconds into the motion. The height of the ball at $t = 1.25$ is:

$$H(1.25) = -16(1.25)^2 + 40(1.25) + 6 = 31.$$

The maximum height of the ball is 31 feet. Here is the graph of $H(t)$:



The graph shows the part of the parabola that is relevant to the motion in our example.

Factored Form of a Quadratic Function

We know already that if we can factor a given quadratic function, finding its zeros (or, equivalently, horizontal intercepts) is very easy. For example, the zeros of the function:

$$f(x) = 2(x - 1)(x + 1.5)$$

are $x = 1$ and $x = -1.5$.

The other way around is also true: if we have zeros, x_1 and x_2, of a given quadratic function $f(x) = ax^2 + bx + c$, we can write the function in factored form:

$$f(x) = a(x - x_1)(x - x_2).$$

Factored Form of a Quadratic Function

Let a quadratic function $f(x) = ax^2 + bx + c$ be given. Let x_1, x_2 be zeros (i.e. roots) of $f(x)$; that is, real numbers such that $f(x_1) = 0$ and $f(x_2) = 0$. Then $f(x)$ can be written in **factored form**:

$$f(x) = a(x - x_1)(x - x_2).$$

If $x_1 = x_2$, x_1 is called a *double root* of $f(x)$.

Note: Every quadratic function can be written in standard form and in vertex form. Not every quadratic function can be written in factored form for real numbers x_1, x_2. If a quadratic function has no real zeros, or equivalently no horizontal intercepts, it cannot be written in factored form.

Example 3.4.5. Find the zeros for a given quadratic function using any method you wish. Express the function in factored form.

(a) $f(x) = 6x^2 - 27x + 12$ (b) $g(x) = x^2 - 2x - 11$ (c) $h(x) = -x^2 - 2$

Solution. (a) To find the zeros of $f(x)$, we have to solve the quadratic equation:

$$6x^2 - 27x + 12 = 0.$$

Notice that $a = 6$, $b = -27$, and $c = 12$. So by the quadratic formula:

$$\begin{aligned} x &= \frac{-(-27) \pm \sqrt{(-27)^2 - 4 \cdot 6 \cdot 12}}{2 \cdot 6} \\ &= \frac{27 \pm \sqrt{441}}{12} \\ &= \frac{27 \pm 21}{12}. \end{aligned}$$

We simplify and obtain the two solutions:

$$x_1 = \frac{27 - 21}{12} = \frac{6}{12} = \frac{1}{2}$$

and

$$x_2 = \frac{27 + 21}{12} = \frac{48}{12} = 4.$$

The function $f(x)$ in factored form is

$$f(x) = 6(x - \frac{1}{2})(x - 4).$$

(b) We set up the equation to find the zeros of $g(x)$:

$$x^2 - 2x - 11 = 0.$$

We have $a = 1$, $b = -2$, and $c = -11$. So by the quadratic formula:

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-11)}}{2 \cdot 1} \\ &= \frac{2 \pm \sqrt{48}}{2} \\ &= \frac{2 \pm \sqrt{16 \cdot 3}}{2} \\ &= \frac{2 \pm 4\sqrt{3}}{2} \\ &= 1 \pm 2\sqrt{3}. \end{aligned}$$

Hence, the two solutions are

$$x_1 = 1 + 2\sqrt{3}$$

and

$$x_2 = 1 - 2\sqrt{3}.$$

The function $g(x)$ in factored form is

$$g(x) = (x - (1 + 2\sqrt{3}))(x - (1 - 2\sqrt{3})).$$

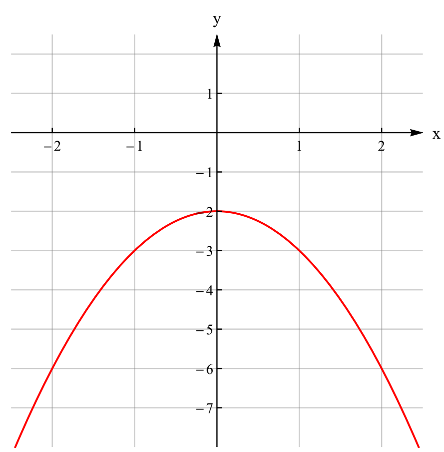
(c) The equation we have to solve is:

$$-x^2 - 2 = 0.$$

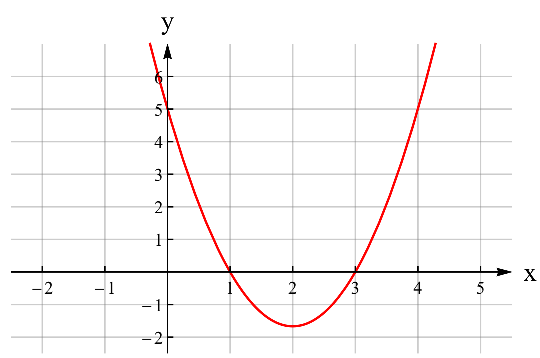
Since $a = -1$, $b = 0$, and $c = -2$, the discriminant is negative: $D = 0^2 - 4(-1)(-2) = -8 < 0$. Hence, the equation has no (real) solutions and $h(x)$ cannot be written in factored form. Of course not. $h(x)$ in vertex form is:

$$h(x) = -1 \cdot (x - 0)^2 - 2.$$

The vertex is below the x-axis at $(0, -2)$ and the parabola opens down. So there cannot be any horizontal intercepts. Here is the graph of $h(x)$:



Example 3.4.6. Use the graph of a quadratic function $y = f(x)$ given below to find a formula for the function in factored form and in standard form.



Solution. The graph gives us the horizontal intercepts or equivalently the zeros of the function $f(x)$: $x = 1$ and $x = 3$. Hence, $f(x)$ in factored form is:

$$f(x) = a(x-1)(x-3).$$

We still don't have the value of the leading coefficient a. To find a, we will use the vertical intercept $y = 5$ of the function. The intercept is clearly seen on the graph. The vertical intercept is the value of the function at $x = 0$. Thus, $f(0) = 5$. We substitute $x = 0$ into factored form of $f(x)$ and obtain:

$$a(0-1)(0-3) = 5.$$

We simplify the equation:

$$a \cdot (-1) \cdot (-3) = 5$$

which gives:

$$a \cdot 3 = 5$$

and finally:

$$a = \frac{5}{3}.$$

The function $f(x)$ in factored form is:

$$f(x) = \frac{5}{3}(x-1)(x-3).$$

To obtain $f(x)$ in standard form, we multiply out all terms and simplify:

$$f(x) = \frac{5}{3}(x-1)(x-3) = \frac{5}{3}(x^2 - 3x - x + 3) = \frac{5}{3}(x^2 - 4x + 3) = \frac{5}{3}x^2 - \frac{20}{3}x + 5.$$

Practice Problems for Section 3.4

In Problems 1-6, find all real solutions to each quadratic equation. Give exact and approximate values rounded off to three decimal places.

1. $x^2 - 5x - 14 = 0$
2. $x(x+3) = -2$
3. $6x^2 + x - 1 = 0$
4. $x^2 - 2x - 2 = 0$
5. $-4x^2 - 17x - 15 = 0$
6. $2x(3-x) = 4(x-5)$

In Problems 7-10, find the intercepts and vertex of each quadratic function. Give exact and approximate values rounded off to three decimal places.

7. $y = x^2 + 2x - 3$
8. $f(x) = x^2 + 8x + 10$
9. $g(x) = 3x^2 - 6x - 1$
10. $y = t^2 - 4t + 5$

In Problems 11-14, calculate the discriminant of each quadratic equation and use it to determine whether the equation will have one real solution, two distinct real solutions, or no real solutions.

11. $3x^2 - 12x + 12 = 0$

12. $8x^2 + 14x + 2 = 0$

13. $3x^2 - 8x + 15 = 0$

14. $x(x-4) = 10$

15. A baseball is thrown straight up with the initial speed of 50 ft/sec by a player who is 6 feet tall. Let $H(t)$ be the height of the ball above the ground t seconds later.

(a) Find a formula for $H(t)$.

(b) When does the ball reach its maximum height and what is its maximum height?

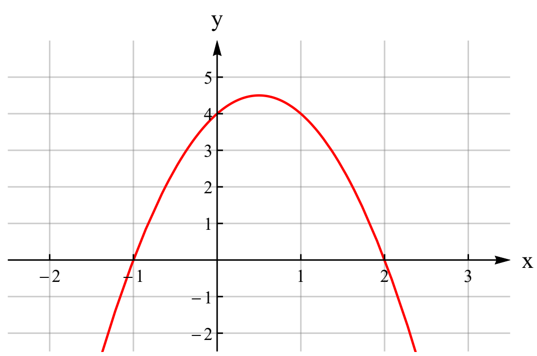
Round off your answers to two decimal places.

In Problems 16-17 rewrite a given quadratic function in factored form. Be sure to use the exact values of the zeros of each function.

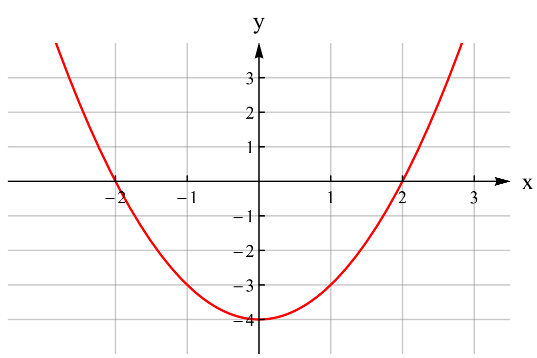
16. $f(x) = 3x^2 - 3x - 6$

17. $g(x) = x^2 + 6x + 7$

18. The graph of a quadratic function $f(x)$ is given below. Write the function in factored form. Then rewrite the function in standard form.



19. The graph of a quadratic function $g(x)$ is given below. Write the function in factored form. Then rewrite the function in standard form.



Chapter 4

Power Functions

4.1 Algebra of Powers: Integral Exponents

In the next two sections we review algebra of power expressions of the form:

$$a^p$$

where a and p are given numbers. The number a is called the *base* in the expression, p is called the *exponent*.

We begin with integral exponents; that is, exponents that are integers.

If p is a positive integer $p = 1, 2, 3, \ldots$, then a^p is simply a short way of writing repeated multiplication:

$$a^p = \underbrace{a \cdot a \cdot \ \ldots \cdot a}_{p\, times}$$

In particular, for $p = 1, 2, 3 \ldots$:

$$0^p = 0, \quad 1^p = 1.$$

We expand this simple definition to the exponent $p = 0$ by defining for any number a:

$$a^0 = 1.$$

In particular, by definition:

$$0^0 = 1.$$

Next, we extend the definition to negative integer exponents by defining for every $a \neq 0$ and every $p = 0, 1, 2, 3, \ldots$

$$a^{-p} = \frac{1}{a^p}$$

In particular:

$$a^{-1} = \frac{1}{a}$$

These simple definitions easily imply the basic properties of power expressions.

Rules of Exponents — Integral Exponents

Let a, b be given numbers, p and r be integers. Then the following equalities hold provided both sides are defined:

1. $a^0 = 1$
2. $a^{-p} = \dfrac{1}{a^p}$
3. $a^p \cdot a^r = a^{p+r}$
4. $\dfrac{a^p}{a^r} = a^{p-r}$
5. $(a^p)^r = a^{p \cdot r}$
6. $(a \cdot b)^p = a^p \cdot b^p$
7. $\left(\dfrac{a}{b}\right)^p = \dfrac{a^p}{b^p}$
8. $\left(\dfrac{a}{b}\right)^{-p} = \left(\dfrac{b}{a}\right)^p$

An expression in the formulas above may not be defined if there is a zero in the denominator. For example, in Property 2, $\frac{1}{a^p}$ is not defined if $a = 0$ and $p \neq 0$. So Property 2 holds for $a \neq 0$.

The properties of exponents listed above follow very easily from our definitions. For example, to illustrate Property 3 observe:

$$a^p \cdot a^r = \underbrace{a \cdot a \cdot \ \dots \cdot a}_{p \ times} \cdot \underbrace{a \cdot a \cdot \ \dots \cdot a}_{r \ times} = \underbrace{a \cdot a \cdot \ \dots \cdot a}_{p+r \ times} = a^{p+r}$$

Example 4.1.1. Rewrite each expression given below as a power of 3; that is, in the form 3^p for some constant p.

(a) $\dfrac{1}{3^2}$ (b) $3^5 \cdot 3^{-2}$ (c) $\dfrac{(3^2)^3}{3^8}$ (d) $\left(\dfrac{3^{-3}}{3^4}\right)^2$ (e) $\left(\dfrac{3^5}{27}\right)^2$

Solution. (a) All we need is Property 2:

$$\frac{1}{3^2} = 3^{-2}.$$

We rewrote the expression as 3^p for $p = -2$.

(b) By Property 3:

$$3^5 \cdot 3^{-2} = 3^{(5+(-2))} = 3^3.$$

(c) By Property 5 and 4:

$$\frac{(3^2)^3}{3^8} = \frac{3^6}{3^8} = 3^{(6-8)} = 3^{-2}.$$

(d) As $3^{-3} = \dfrac{1}{3^3}$ we have:

$$\left(\frac{3^{-3}}{3^4}\right)^2 = \left(\frac{1}{3^3 \cdot 3^4}\right)^2 = \left(\frac{1}{3^7}\right)^2 = (3^{-7})^2 = 3^{-14}.$$

(e) Note that $27 = 3^3$. Using Property 7:

$$\left(\frac{3^5}{27}\right)^2 = \left(\frac{3^5}{3^3}\right)^2 = \frac{3^{10}}{3^6} = 3^4.$$

Example 4.1.2. Use Rules of Exponents to simplify the following expressions if possible:

(a) $(x^2 \cdot x^3)^2$ (b) $\left(\dfrac{x^2y^5}{x^4}\right)^{-3}$ (c) $\left(\dfrac{x^{-2}y^5}{x^{-3}y^2}\right)^{-1}$ (d) $(a+b)^7$

Solution. (a) We use Property 3 to simplify the expression under the outside power 2. Then we use Property 5:

$$(x^2 \cdot x^3)^2 = (x^5)^2 = x^{10}.$$

(b) Let's begin by simplifying the expression under the power -3 using Property 4:

$$\left(\frac{x^2y^5}{x^4}\right)^{-3} = \left(\frac{y^5}{x^2}\right)^{-3}.$$

Now we use Property 8 and then Property 7:

$$\left(\frac{y^5}{x^2}\right)^{-3} = \left(\frac{x^2}{y^5}\right)^3 = \frac{(x^2)^3}{(y^5)^3} = \frac{x^6}{y^{15}}.$$

(c) Let's simplify the expression under the power -1 using Property 2:

$$\left(\frac{x^{-2}y^5}{x^{-3}y^2}\right)^{-1} = \left(\frac{x^3y^5}{x^2y^2}\right)^{-1}.$$

By Property 4 and then 2:

$$\left(\frac{x^3y^5}{x^2y^2}\right)^{-1} = (xy^3)^{-1} = \frac{1}{xy^3}.$$

(d) There is no rule for the power of a sum or a difference! We cannot simplify $(a+b)^7$ using Rules of Exponents. You certainly **cannot** distribute the power 7 and write the expression as $a^7 + b^7$.

Practice Problems for Section 4.1

In Problems 1-4, rewrite a given expression as a power of 2; that is, express it in the form 2^m for some m.

1. $\dfrac{4}{2^4}$

2. $(2^3)^{-2}$

3. $\dfrac{8^2}{2}$

4. $\left(\dfrac{2^3}{2^4 \cdot 2^5}\right)^2$

In Problems 5-8, rewrite a given expression as a power of 5; that is, express it in the form 5^m for some m.

5. $5^2 \cdot 5^{-3}$

6. $\left(\dfrac{25^2}{5}\right)^{-1}$

7. $\left(\dfrac{5^{-1}}{5^{-2}}\right)^2$

8. $\left(\dfrac{5^3}{25^2}\right)^{-2}$

In Problems 9-15 simplify a given expression if possible. If not possible, state so.

9. $(x^2y^5)^0$

10. $\left(\dfrac{x^{-3}}{xy^2}\right)^{-1}$

11. $\dfrac{(x-y)^3}{x}$

12. $(2x^2y^4)^3$

13. $\dfrac{x+y}{y}$

14. $\left(\dfrac{\pi(x^2)^6}{x^2y^{-4}}\right)^2$

15. $\dfrac{x^2+y^3}{xy}$

16. Let p and r be positive integers. Use the definition of exponentiation:

$$a^m = \underbrace{a \cdot a \cdot \ \dots \cdot a}_{m\ times}$$

to explain why the following formula is valid:

$$(a^p)^r = a^{p \cdot r}$$

4.2 Algebra of Powers: Fractional Exponents

Roots and Radicals

Let's review the concepts of roots and radicals. Please note that in this book we stay within real numbers and we do not consider complex numbers. In particular, when we talk about roots, we mean real roots.

We say that a number y is a root of order 2 (a square root) of a number a if $y^2 = a$. We say that a number y is a root of order 3 (a cubic root) of a if $y^3 = a$ and so on. Here is the definition.

Definition of Roots

Let a be a given number. Let n be a positive integer. We say that y is a root of order n of a, or an nth root of a, if $y^n = a$.

For example, let $a = 4$, $n = 2$. The number 4 has two roots of order 2: $y = 2$ and $y = -2$. Of course, $2^2 = (-2)^2 = 4$. Take $a = -4$ and $n = 2$. The number -4 has no roots of order 2. Indeed, for every number y the square of it y^2 is positive or zero. It cannot be negative so it will never equal -4. We say that the square root of -4 does not exist or is undefined.

Take $a = -64$ and $n = 3$. The number -64 has exactly one root of order 3 which is $y = -4$. Indeed, $(-4)^3 = -64$. Observe that $4^3 = 64$ so 4 is not a cubic root of -64.

Roots and Radicals

Let a be a given number. Let n be a positive integer.

- Let n be even.
  - If $a < 0$, then a has no nth order roots.
  - If $a > 0$, then a has two nth roots, one positive and one of the same magnitude but negative. We denote the **positive** nth root as:

    $$\sqrt[n]{a}.$$

    (We use the "radical" symbol $\sqrt{\ }$.) The two nth roots can then be written as

    $$\sqrt[n]{a} \quad \text{and} \quad -\sqrt[n]{a}$$

    where $\sqrt[n]{a}$ is the positive root.
- Let n be odd. Then a has exactly one nth root which we denote as

  $$\sqrt[n]{a}.$$

  If a is negative, the root is negative; if a is positive the root is positive.
- If $a = 0$, then a has one root of any order: $\sqrt[n]{0} = 0$.

Note: For radicals of order 2, we use the notation:

$$\sqrt[2]{a} = \sqrt{a}.$$

Example 4.2.1. Find all (real) roots specified below and write them in terms of radicals.

(a) All roots of order 4 of 81 (c) All roots of order 3 of -27

(b) All roots of order 2 of 7 (d) All roots of order 4 of -81

Solution. (a) We are looking for all numbers y such that $y^4 = 81$. The order, 4, is even. Hence we have two roots, one positive and one negative, the opposite of each other. The positive root is denoted by $\sqrt[4]{81}$ and the two roots are:

$$\sqrt[4]{81} \quad \text{and} \quad -\sqrt[4]{81}.$$

The positive number whose 4th power is 81 is 3. That is:

$$\sqrt[4]{81} = 3.$$

Hence, the two roots of order 4 of 81 are 3 and -3. Indeed, $3^4 = (-3)^4 = 81$.

(b) 7 has two roots of order 2: $\sqrt{7}$ and $-\sqrt{7}$. We cannot easily guess them as they are not integers. We can use our calculator, though, and calculate $\sqrt{7} \approx 2.65$. The two roots are then approximately 2.65 and -2.65.

(c) The order, 3, is odd. Hence, there is only one root of order 3 of -27 denoted as $\sqrt[3]{-27}$. As $(-3)^3 = -27$, we have

$$\sqrt[3]{-27} = -3.$$

(d) A negative number $a = -81$ has no roots of even orders and 4 is even. Hence, no roots.

By definition, x is a root of order n of a if x is a solution to the equation:

$$x^n = a$$

Hence, roots and radicals appear naturally when solving equations containing powers of the unknown. We saw plenty of radicals in Chapter 3 in the context of quadratic equations.

Example 4.2.2. Solve for x. Find all (real) solutions and give their exact as well as approximate values.

(a) $2x^2 - 5 = 7$ (b) $-3x^3 = 27$ (c) $2x^4 + 3 = 4$ (d) $2(x-1)^2 + 3 = 11$

Solution. (a) We add 5 to both sides of the equation and then divide both sides by 2. We obtain:

$$x^2 = 6.$$

Solutions to the equation are roots of order 2 of 6. There are two such roots:

$$x = \sqrt{6} \quad \text{and} \quad x = -\sqrt{6}.$$

We use a calculator and obtain approximate values to three decimal places:

$$x = \sqrt{6} \approx 2.449 \quad \text{and} \quad x = -\sqrt{6} \approx -2.449.$$

(b) We divide both sides of the equation by -3 and obtain:

$$x^3 = -9.$$

There is one cubic root of -9. Hence, our equation has one solution:

$$x = \sqrt[3]{-9} \approx -2.08.$$

(c) We subtract 3 from both sides of the equation, then divide by 2. The equation becomes:

$$x^4 = 0.5.$$

We are looking for all roots of order 4 of 0.5. As 4 is even, we have two such roots:

$$x = \sqrt[4]{0.5} \quad \text{and} \quad x = -\sqrt[4]{0.5}.$$

We use a calculator and find approximate values of the solutions:

$$x = \sqrt[4]{0.5} \approx 0.841 \quad \text{and} \quad x = -\sqrt[4]{0.5} \approx -0.841.$$

(d) We begin by "solving" for $(x-1)$. Then we will isolate x. We subtract 3 from both sides and divide both sides by 2. The equation becomes:

$$(x-1)^2 = 4.$$

Hence, $(x-1)$ is a root of order 2 of 4. There are two such roots:

$$(x-1) = \sqrt{4} \quad \text{and} \quad (x-1) = -\sqrt{4}.$$

This gives:

$$x - 1 = 2 \quad \text{and} \quad x - 1 = -2.$$

We solve each of the two equations for x and obtain two solutions:

$$x = 3 \quad \text{and} \quad x = -1.$$

Here are a few important properties of roots.

Properties of Radicals

Let a, b be given numbers. Let n, m be a positive integers. Then the following equalities hold provided that the roots involved exist and both sides are defined:

- $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
- $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$
- $\sqrt[n]{a^n} = |a|$ if n is even
- $\sqrt[n]{a^n} = a$ if n is odd

Powers with Fractional Exponents and Arbitrary Exponents

So far, we have defined powers a^p for all **integer** exponents p.

To extend the definition to fractional exponents: $a^{\frac{m}{n}}$, we will use roots. In the first step, for every positive integer n, we define $a^{\frac{1}{n}}$ as:

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

The obvious logic behind such a definition is the fact that from the definition of a root:

$$(\sqrt[n]{a})^n = a$$

Hence:

$$(a^{\frac{1}{n}})^n = a$$

which is what Rules of Exponents would dictate. In the next step, we define $a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = (\sqrt[n]{a})^m$ which seems to make sense. Here is a precise definition of a power with a fractional exponent:

Fractional Exponents

Let a be a given number. Let m and n be a positive integers. Assume that $\sqrt[n]{a}$ exists. We define:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

$$a^{-\frac{m}{n}} = \frac{1}{a^{m/n}} \quad \text{provided} \quad a \neq 0$$

Note that $\sqrt[n]{a}$ exists unless n is even and a is negative. The combination of negative radicands (numbers under radicals) and even roots and powers cause a lot of possible problems with the behavior of fractional exponents and Rules of Exponents are not always satisfied. For example, we would expect that the equality

$$\left(a^{\frac{1}{n}}\right)^m = (a^m)^{\frac{1}{n}}$$

always holds. But with negative bases combined with even roots and powers it is not always so. Just as an example, consider:

$$\left((-4)^{\frac{1}{4}}\right)^2 \quad \text{and} \quad \left((-4)^2\right)^{\frac{1}{4}}.$$

$\left((-4)^{\frac{1}{4}}\right)^2 = (\sqrt[4]{-4})^2$ is undefined as $\sqrt[4]{-4}$ is undefined. But

$$\left((-4)^2\right)^{\frac{1}{4}} = 16^{\frac{1}{4}} = \sqrt[4]{16} = 2.$$

Therefore, when we talk about fractional powers, we will most often assume that bases are positive except for some very simple cases like:

$$a^{\frac{1}{3}} = \sqrt[3]{a}.$$

Example 4.2.3. Find:

(a) $4^{\frac{1}{2}}$ (b) $(-4)^{\frac{1}{2}}$ (c) $27^{\frac{1}{3}}$ (d) $(-27)^{\frac{1}{3}}$ (e) $(1700)^{\frac{1}{20}}$.

Solution. (a) By definition $4^{\frac{1}{2}} = \sqrt[2]{4} = \sqrt{4} = 2$. Note that $4^{\frac{1}{2}}$ is the **positive** of the two square roots of 4 as is $\sqrt{4}$.

(b) $(-4)^{\frac{1}{2}} = \sqrt{-4}$ is undefined. An even-order root of a negative number is undefined. Hence, $(-4)^{\frac{1}{2}}$ is undefined.

(c) $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$.

(d) $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$.

(e) $(1700)^{\frac{1}{20}} = \sqrt[20]{1700} \approx 1.451$.

Depending on your calculator, it may be easier to calculate a fractional power than to enter the corresponding radical. You can simply enter: $1700 \wedge (1/20)$.

We have defined powers a^p for integer and fractional exponents p. As you may know, not all real numbers can be expressed as fractions — irrational numbers cannot be expressed as fractions. Can we define powers a^p for all real numbers p? The answer is affirmative provided the base a is positive. We will skip the construction that is long and involved. It suffices to know that a^p can be defined for all exponents and Rules of Exponents are preserved.

Rules of Exponents — Arbitrary Exponents

Let a, b be given **positive** numbers, p and r be real numbers. Then the following equalities hold:

1. $a^{-p} = \dfrac{1}{a^p}$
2. $a^p \cdot a^r = a^{p+r}$
3. $\dfrac{a^p}{a^r} = a^{p-r}$
4. $(a^p)^r = a^{p \cdot r}$
5. $(a \cdot b)^p = a^p \cdot b^p$
6. $\left(\dfrac{a}{b}\right)^p = \dfrac{a^p}{b^p}$
7. $\left(\dfrac{a}{b}\right)^{-p} = \left(\dfrac{b}{a}\right)^p$

Example 4.2.4. Rewrite each of the following expressions as a power of 5; that is, in the form 5^p for some p.

(a) $\sqrt[3]{5}$ (b) $5\sqrt{5}$ (c) $\dfrac{5\sqrt{5}}{25}$ (d) $\left(\dfrac{5}{\sqrt[3]{5}}\right)^2$

Solution. (a) By definition of fractional powers:

$$\sqrt[3]{5} = 5^{\frac{1}{3}}.$$

(b) By definition of fractional powers and Property 2:

$$5\sqrt{5} = 5 \cdot 5^{\frac{1}{2}} = 5^{1+\frac{1}{2}} = 5^{\frac{3}{2}}.$$

(c) Note that $25 = 5^2$. Using Properties 2 and 3:

$$\frac{5\sqrt{5}}{25} = \frac{5 \cdot 5^{\frac{1}{2}}}{5^2} = \frac{5^{\frac{3}{2}}}{5^2} = 5^{\frac{3}{2}-2} = 5^{-\frac{1}{2}}.$$

(d) Note that $\sqrt[3]{5} = 5^{\frac{1}{3}}$. Using Rules of Exponents we get:

$$\left(\frac{5}{\sqrt[3]{5}}\right)^2 = \left(\frac{5}{5^{\frac{1}{3}}}\right)^2 = \left(5^{\frac{2}{3}}\right)^2 = 5^{\frac{2}{3}\cdot 2} = 5^{\frac{4}{3}}.$$

Example 4.2.5. Use Rules of Exponents to simplify the following expressions. Write your answers in terms of powers and not in terms of radicals. We assume that a, b, x, and y are all positive.

(a) $\sqrt{\dfrac{36}{a^5}}$ (b) $\sqrt[3]{8x^2y^6}$ (c) $\dfrac{b^{\frac{3}{2}}\sqrt{a^3}}{ba^2}$ (d) $\dfrac{14x^{-2}}{21\sqrt{x^7}}$

Solution. (a)

$$\sqrt{\frac{36}{a^5}} = \frac{\sqrt{36}}{\sqrt{a^5}} = \frac{6}{a^{5/2}} = 6a^{-(5/2)}.$$

(b)

$$\sqrt[3]{8x^2y^6} = \sqrt[3]{8} \cdot \sqrt[3]{x^2} \cdot \sqrt[3]{y^6} = 2 \cdot (x^2)^{1/3} \cdot (y^6)^{1/3} = 2 \cdot x^{2/3} \cdot y^{6/3} = 2x^{2/3}y^2.$$

(c)

$$\frac{b^{\frac{3}{2}}\sqrt{a^3}}{ba^2} = \frac{b^{\frac{3}{2}}a^{\frac{3}{2}}}{ba^2} = b^{\frac{3}{2}-1}a^{\frac{3}{2}-2} = b^{1/2}a^{-(1/2)}.$$

(d) We have $14 = 2 \cdot 7$, $21 = 3 \cdot 7$, $x^{-2} = \dfrac{1}{x^2}$, $\sqrt{x^7} = x^{7/2}$. Hence:

$$\frac{14x^{-2}}{21\sqrt{x^7}} = \frac{2}{3x^2x^{7/2}} = \frac{2}{3}x^{-(11/2)}.$$

Practice Problems for Section 4.2

In Problems 1-6 solve an equation for x or for b. Be sure to list all solutions. Give exact and approximate values rounded off to three decimal places. If there are no solutions say so.

1. $\dfrac{2}{5}x^4 + 3 = 11$
2. $1500b^{30} = 17000$
3. $5x^3 - 6 = -8$
4. $\dfrac{5}{3}x^6 + 9 = 4$
5. $3(x-1)^2 + 1 = 10$
6. $(x+2)^3 = -8$

In Problems 7-10 rewrite an expression as a power of 3; that is, in the form 3^r for some r. If it is not possible, say so.

7. $\dfrac{9^2}{3\sqrt{3}}$
8. $\dfrac{27}{\sqrt{3}}$
9. $\dfrac{3^2\sqrt[3]{9}}{\sqrt{3}}$
10. $\dfrac{3^{1/5}\sqrt[4]{3}}{\sqrt{3}}$

In Problems 11-14 simplify a given expression. If it is not possible, say so. x and y are assumed to be positive.

11. $\dfrac{\sqrt{x}+\sqrt{y}}{xy}$

12. $(\sqrt[4]{y}\sqrt[3]{x})^4$

13. $\dfrac{x^{\frac{2}{3}}y^3}{xy}$

14. $\dfrac{x^2y^3}{\sqrt{x}+\sqrt[3]{y}}$

4.3 Power Functions: Positive Integral Exponents

In the next two sections we look at the properties and the graphs of the so-called power functions.

Power Functions

A function $y = f(x)$ is called a **power function** if $f(x)$ can be expressed in the form:

$$f(x) = kx^p$$

where k and p are constants, $k \neq 0$. k is called the coefficient of the power function $f(x)$ and p is called the exponent.

Note: When y depends on x according to the formula $y = kx^p$ we say that y is **directly proportional** (or proportional) to x^p with the coefficient of proportionality k. If $y = \dfrac{k}{x^p}$, we say that y is **inversely proportional** to x^p with the coefficient of proportionality k. So power functions express proportionality of the dependent variable to powers of the independent variable.

As the title suggests in this section we look at the case when the exponent p is a positive integer.

Example 4.3.1. Which of the functions below are power functions? For those which are, rewrite in the standard form $y = kx^p$. Identify the coefficient and the exponent.

(a) $f(x) = 2x^2 \cdot 4x^3$ (b) $g(x) = (-(x^3))^5$ (c) $h(x) = \dfrac{(x^2)^4}{7x^3}$

(d) $r(x) = 2x^3 + 7x^2$ (e) $m(x) = \dfrac{8x}{\pi^2}$ (f) $l(x) = 3 \cdot 2^x$.

Solution. (a) The function $f(x)$ is a power function. Using Rules of Exponents from Section 4.1, we can rewrite:

$$f(x) = 2x^2 \cdot 4x^3 = (2 \cdot 4)(x^2x^3) = 8x^5.$$

The coefficient $k = 8$, the exponent $p = 5$.

(b) $g(x)$ is a power function:

$$g(x) = (-(x^3))^5 = (-1 \cdot (x^3))^5 = (-1)^5(x^3)^5 = -1 \cdot x^{15} = -x^{15}.$$

The coefficient $k = -1$, the exponent $p = 15$.

(c) $h(x)$ is a power function as well:

$$h(x) = \frac{(x^2)^4}{7x^3} = \frac{x^8}{7x^3} = \frac{1}{7}x^{8-3} = \frac{1}{7}x^5.$$

The coefficient $k = \frac{1}{7}$, the exponent $p = 5$.

(d) The function $r(x)$ is not a power function. It cannot be rewritten as $r(x) = kx^p$. The function $r(x)$ is a sum of two power functions, $2x^3$ and $7x^2$. A function that is a sum of power functions is called a polynomial function.

(e) The function $m(x)$ is a power function. Remember that π is just a constant.

$$m(x) = \frac{8x}{\pi^2} = \frac{8}{\pi^2}x.$$

The coefficient $k = \frac{8}{\pi^2}$, the exponent $p = 1$.

(f) $l(x)$ is not a power function. Observe that in $l(x)$ the base of the power expression 2^x is constant and equal to 2. The exponent x is a variable. In a power function, it is the other way around: the base is a variable and the exponent is constant.

Example 4.3.2. A ball dropped from the Empire State Building has traveled down the distance of $d(t)$ feet after t seconds where:

$$d(t) = 16t^2.$$

(a) Is the function $d(t)$ a power function? If yes, identify the exponent and the coefficient.

(b) The Empire State Building is 1250 ft tall. How long will it take for the ball to hit the ground?

Solution. (a) $d(t)$ is a power function with $k = 16$ and $p = 2$. ($d(t)$ is also a quadratic function with $a = 16$, $b = 0$, and $c = 0$.)

(b) The ball will hit the ground when it has traveled 1250 feet. That is, for a positive t such that:

$$d(t) = 16t^2 = 1250.$$

To solve the equation for t we divide both sides by 16:

$$t^2 = \frac{1250}{16}.$$

The two solutions to this quadratic equation are:

$$t = \pm\sqrt{\frac{1250}{16}}.$$

Since t has to be positive:

$$t = \sqrt{\frac{1250}{16}}.$$

Hence, the ball will hit the ground after:

$$t = \sqrt{\frac{1250}{16}} \approx 8.8 \text{ seconds.}$$

We can calculate the approximate answer by using the radical button on your calculator. As we learned in the previous section:

$$\sqrt{\frac{1250}{16}} = \left(\frac{1250}{16}\right)^{\frac{1}{2}} \approx 8.8.$$

So we can calculate the power $\frac{1}{2}$ instead. This is a useful observation in case we have to calculate radicals of orders other than 2.

Note that in terms of proportionality, we can say that the distance d is directly proportional to t^2 with the coefficient of proportionality 16.

Example 4.3.3. Let $V(r)$ be the volume of a sphere of radius r. We know from elementary geometry that:

$$V(r) = \frac{4}{3}\pi r^3$$

(a) Is $V(r)$ a power function? If yes, find the coefficient and the exponent.

(b) What radius is required for the volume to be 25 cm^3?

Solution. Yes, $V(r)$ is a power function. The coefficient k is $\frac{4\pi}{3}$, the exponent p is 3.

(b) We are looking for the radius r, in centimeters, such that:

$$\frac{4}{3}\pi r^3 = 25$$

We divide both sides by 4 and by π and multiply by 3. The equation becomes:

$$r^3 = \frac{3}{4\pi}25$$

Now we take the power $\frac{1}{3}$ of both sides (in other words, the cubic root) and obtain:

$$(r^3)^{\frac{1}{3}} = \left(\frac{3}{4\pi}25\right)^{\frac{1}{3}}$$

We have $(r^3)^{\frac{1}{3}} = r$. We calculate the right-hand side using our calculator:

$$r = \left(\frac{3}{4\pi}25\right)^{\frac{1}{3}} \approx 1.814 \text{ cm.}$$

The volume is 25 cm^3 when the radius is 1.814 cm.

Graphs of Power Functions: Positive Integral Exponents

Graphs of power functions $y = kx^p$ with exponents p that are positive integers are different for p even and for p odd.

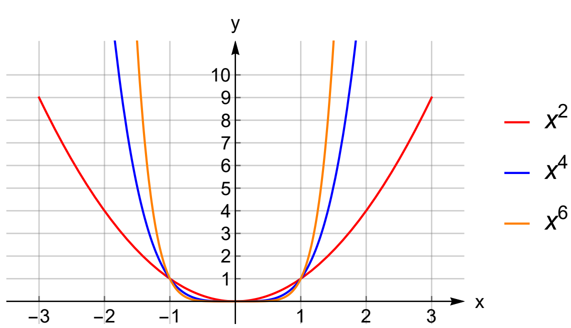
Even Positive Exponents

If p is even, then x^p is positive for all x except for $x = 0$ where $x^p = 0$. Take, for example, $p = 2$:

$1^2 = (-1)^2 = 1, \qquad 2^2 = (-2)^2 = 4, \qquad 3^2 = (-3)^2 = 9, \qquad 4^2 = (-4)^2 = 16$ etc.

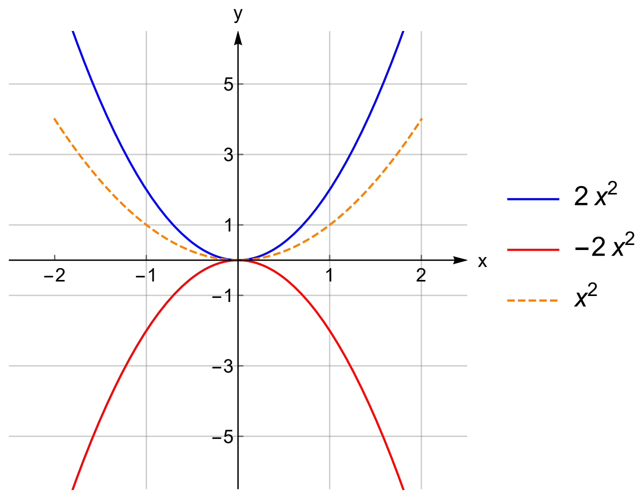
Hence, the graph of the power function $f(x) = kx^2$ as well as $f(x) = kx^p$ for any even p is symmetric about the y-axis as $f(-x) = f(x)$.

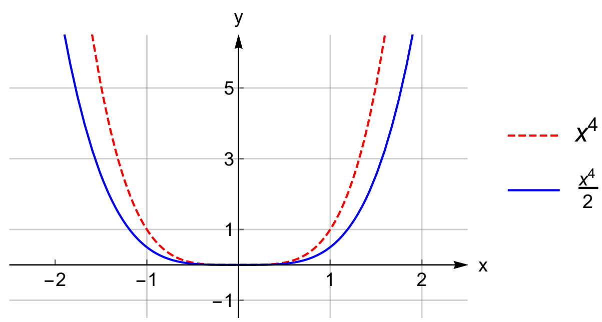
Here are the graphs of the functions $y = kx^p$ for $p = 2, 4, 6$ and $k = 1$:

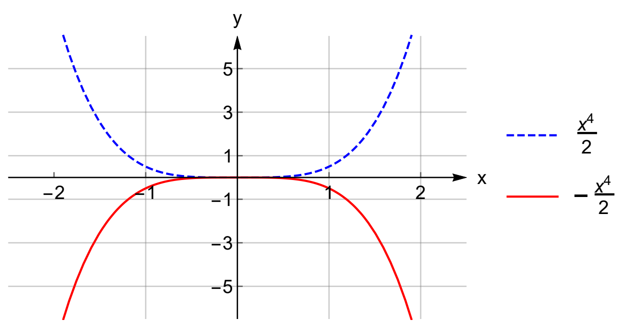


All graphs are U-shaped and reminiscent of the quadratic parabola $y = x^2$.

The coefficient k in $y = kx^p$, stretches or shrinks the graph of $y = x^p$ vertically. Additionally, if $k < 0$, the graph is reflected over the x-axis. Here are some examples:







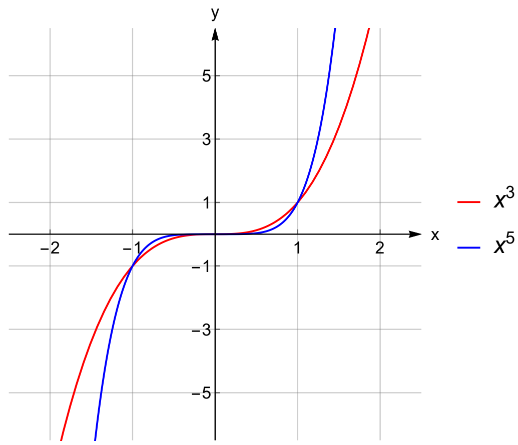
Odd Positive Exponents

If p is odd, then x^p is positive for $x > 0$, it is 0 for $x = 0$, and negative for $x < 0$. Take, for example, $p = 3$:

$$1^3 = 1, \qquad (-1)^3 = -1, \qquad 2^3 = 8, \qquad (-2)^3 = -8, \qquad 3^3 = 27, \qquad (-3)^3 = -27 \text{ etc.}$$

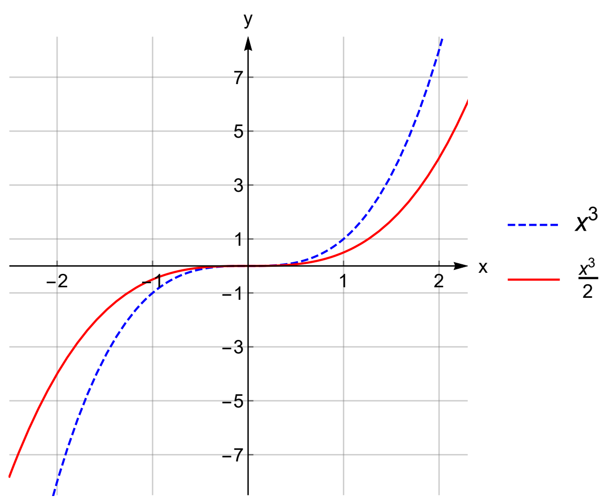
Hence, the graph of the power function $f(x) = kx^3$ as well as $f(x) = kx^p$ for any odd p is symmetric about the origin $(0, 0)$ as $f(-x) = -f(x)$.

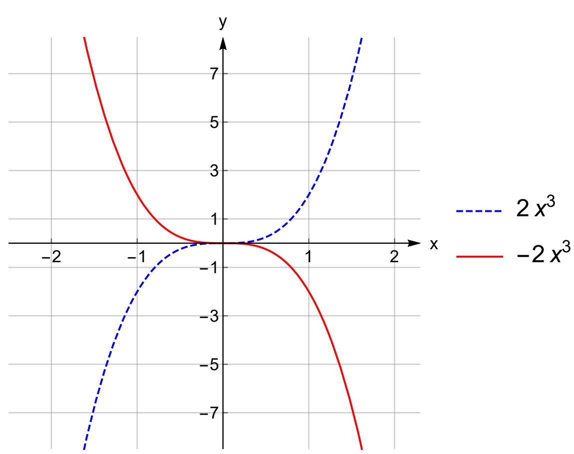
Here are the graphs of the functions $y = kx^p$ for $p = 3, 5$ and $k = 1$:



These time, the graphs are S-shaped.

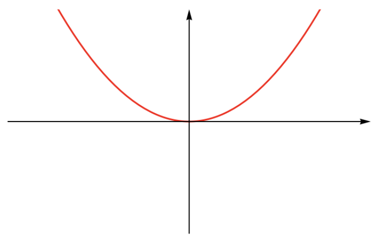
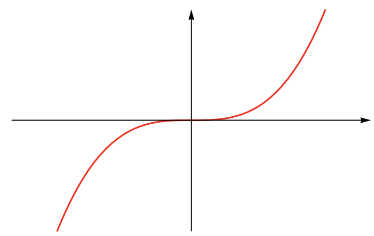
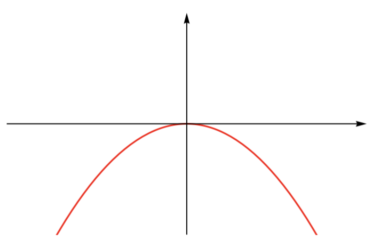
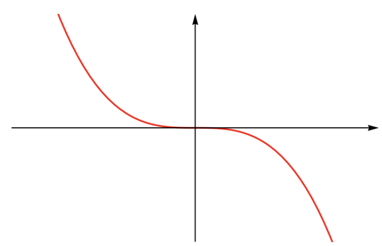
The coefficient k in $y = kx^p$, stretches or shrinks the graph of $y = x^p$ vertically. Additionally, if $k < 0$, the graph is reflected over the x-axis. Here are some examples:





Summary

Here is a summary of how graphs of power functions look for positive integer exponents p, even and odd, and for coefficients k positive and negative:

$f(x) = kx^p$ p positive integer	Even Exponent: p even	Odd Exponent: p odd
Positive Coefficient $k > 0$		
Negative Coefficient $k < 0$		

Practice Problems for Section 4.3

In Problems 1-6, determine whether a given function is a power function. If yes, identify the coefficient k and the exponent p. If not, say so.

1. $y = \dfrac{0.8x^4}{0.2x}$

2. $y = \dfrac{\sqrt{9x^8}}{x}$

3. $f(x) = \dfrac{1}{3} \cdot 3^{x+1}$

4. $g(x) = \sqrt{0.01x^6}$

5. $h(x) = 0.01x^3 - 2x^2$

6. $y = 9\sqrt{\dfrac{x^5}{x}}$

"Braking distance" or "stopping distance" refers to the distance a car will travel from the point when its brakes are fully applied to when it comes to a complete stop[1]. The braking distance is

[1]http://hyperphysics.phy-astr.gsu.edu/hbase/crstp.html, accessed: 7/5/20

proportional to the square of the car's speed and it depends on the coefficient of friction, μ, between the tires and the road surface. Let D denote distance, in feet, and S speed in mph. The formula for the braking distance is:

$$D = \frac{0.034}{\mu} S^2$$

Note that the braking distance does not include a driver's reaction time[2].

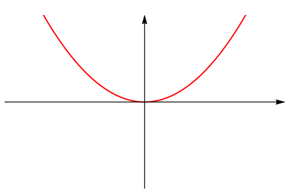
7. Is the braking distance a power function of speed? If yes, give the coefficient k and the exponent p. Assume that μ is a given constant.

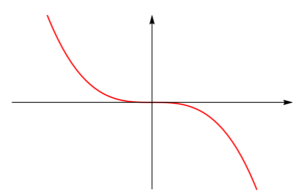
8. The coefficient of friction under normal conditions when the road is dry is $\mu = 0.7$. What is the braking distance of a car that travels on a dry road at 35 mph? What is the braking distance at 70 mph?

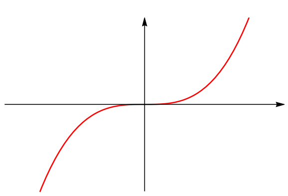
9. By what factor does the braking distance increase when speed S doubles?

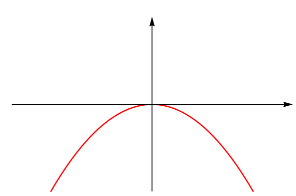
10. The coefficient of friction on a wet road is $\mu = 0.4$. Calculate the braking distance of a car traveling at 70 mph on a wet road.

In Problems 11-14 you see the graph of a power function $y = kx^p$ where p is a positive integer. In each of the graphs, is the exponent p even or odd? Is the coefficient k positive or negative?

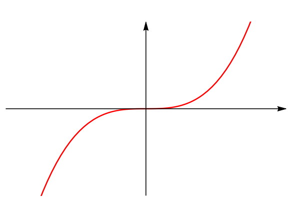
11. 

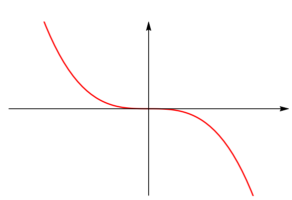
13. 

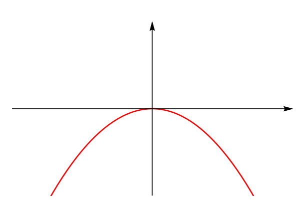
12. 

14. 

15. Which of the graphs below is the graph of the function $f(x) = -4x^5$?

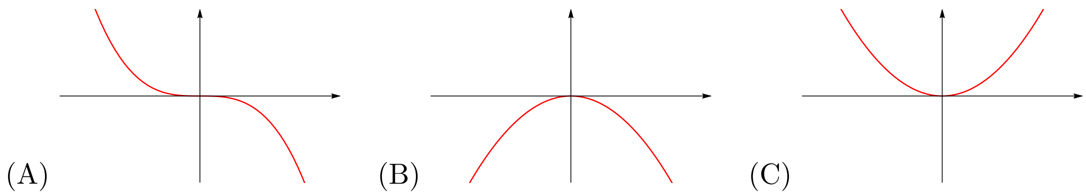
(A) 

(B) 

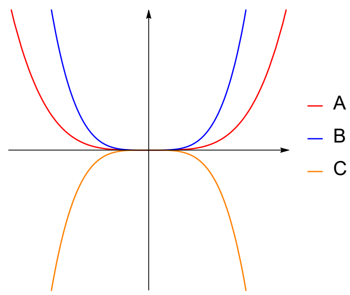
(C) 

[2]https://en.wikipedia.org/wiki/Braking_distance, accessed: 7/5/20

16. Which of the graphs below is the graph of the function $f(x) = -3x^4$?



17. Below you see graphs of the functions $y = -x^4$, $y = \frac{1}{4}x^4$, and $y = x^4$. Decide which is which.



18. The area, A, of a unilateral triangle whose sides have length a is given by:

$$A = \frac{\sqrt{3}}{4}a^2$$

Find the side length a, in cm, which gives the area A equal to 8 cm^2. Round off your answer to three decimal places.

4.4 Power Functions: Fractional and Negative Integral Exponents

In the previous section, we considered power functions $f(x) = kx^p$ in the case when p is a positive integer. When p is a negative integer, power functions behave quite differently.

Negative Integral Exponents

Consider a power function:

$$f(x) = kx^p$$

in the case when the exponent p is a negative integer. The behavior and graphs of such functions are very different than the behavior of power functions for positive integral exponents. Take $k = 1$ and $p = -2$ and $p = -3$ to get an idea. The corresponding power functions are:

$$f(x) = x^{-2} = \frac{1}{x^2}, \quad g(x) = x^{-3} = \frac{1}{x^3}$$

A negative exponent x^{-m} means the reciprocal $\frac{1}{x^m}$. So now the independent variable is in the denominator. The domain of a power function in this case is no longer the set of all real numbers

x as x cannot be 0, $x \neq 0$. As we will soon see, the graphs of power functions with exponents that are negative integers look very differently than in the case of positive integral exponents.

Example 4.4.1. Which of the functions below are power functions? Rewrite those which are in standard form $y = kx^p$. Identify the coefficient and the exponent.

(a) $f(x) = \dfrac{3x^2}{2x^3}$ (b) $g(x) = \dfrac{-\pi}{x^4}$ (c) $h(x) = \dfrac{70}{3^x}$.

Solution. (a) The function $f(x)$ is a power function. Using Rules of Exponents from Section 4.1, we can rewrite:

$$f(x) = \frac{3x^2}{2x^3} = \frac{3}{2x} = \frac{3}{2} \cdot \frac{1}{x} = \frac{3}{2} \cdot x^{-1}.$$

The coefficient $k = \frac{3}{2}$, the exponent $p = -1$.

(b) $g(x)$ is a power function:

$$g(x) = \frac{-\pi}{x^4} = -\pi x^{-4}.$$

The coefficient $k = -\pi$, the exponent $p = -4$.

(c) $h(x)$ is not a power function. Using rules of exponents for arbitrary exponents, we can rewrite:

$$h(x) = \frac{70}{3^x} = 70 \cdot 3^{-x}.$$

The function $h(x) = 70 \cdot 3^{-x}$ is not a power function. Indeed, the base of the power expression 3^{-x} is constant and equal to 3. The exponent $-x$ varies. In a power function, it is the other way around: the base is a variable and the exponent is constant. The function $h(x) = 70 \cdot 3^{-x}$ belongs to an important family of functions called exponential functions. We will study exponential functions in Chapter 5.

Example 4.4.2. Body mass index (BMI) is an easy screening method for weight category — underweight, healthy weight, overweight, and obesity[3]. BMI is calculated as follows:

$$\text{BMI} = \frac{(\text{weight in lb}) \cdot 703}{(\text{height in inches})^2}$$

A person who weighs 170 lb may be underweight or obese— it depends on the person's height. Let $I(h)$ denote body mass index of a person who weighs 170 lb and whose height is h. Then, according to the formula:

$$I(h) = \frac{170 \cdot 703}{h^2}$$

Note that with the weight fixed at 170 lb, body mass index is a power function of height: $I = 119510 \cdot h^{-2}$. (In terms of proportionality, BMI is inversely proportional to the square of

[3]https://www.cdc.gov/healthyweight/assessing/bmi/index.html, accessed: 6/26/20

height.) The obese category is defined as BMI of 30.0 or above. The normal weight category corresponds to BMI between 18.5 and 24.9. Find the height h at and below which a person weighing 170 lb is obese.

Solution. We want to find h such that $I(h) = 30$:

$$\frac{170 \cdot 703}{h^2} = 30$$

We multiply both sides by h^2 and divide by 30. The equation becomes:

$$h^2 = \frac{170 \cdot 703}{30}$$

which gives:

$$h = \pm\sqrt{\frac{170 \cdot 703}{30}}$$

Since h in our problem has to be positive, the solution is:

$$h = \sqrt{\frac{170 \cdot 703}{30}} \approx 63$$

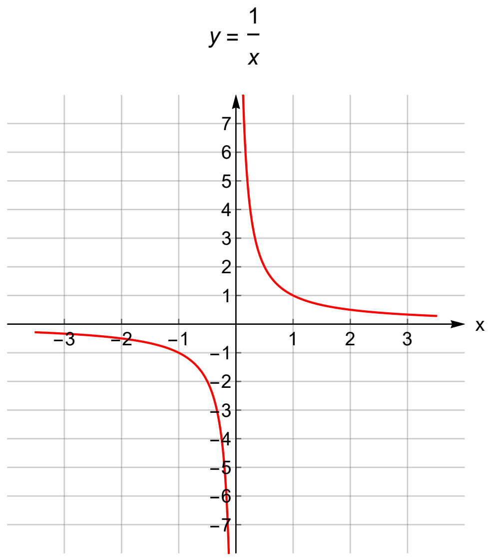
At the height $h = 63$ inches and below a person weighing 170 lb is obese. Note that when h increases $I(h)$ decreases as the denominator of the expression $\frac{170 \cdot 703}{h^2}$ becomes larger. On the other hand, when h gets smaller, $I(h)$ gets larger. Hence, every person shorter that 63 inches who weighs 170 lb is also obese.

Graphs of Power Functions: Negative Integral Exponents

Graphs of power functions $y = kx^p$ with exponents p that are negative integers have a different shape depending on whether p is even or p is odd.

Odd Negative Exponents

Let's begin with the case $k = 1$ and $p = -1$. The graph of the function $y = x^{-1} = \frac{1}{x}$ looks as follows:



The shape of the graph is easily explained by the following table of values:

x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
$y=\frac{1}{x}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-4	undefined	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$

Observe that when x gets larger and larger, say:

$$x = 1,\, 2,\, 3,\, 4,\, 5,\, ...1000\,...,$$

the values $y = \dfrac{1}{x}$ become very close to 0. Indeed, they are:

$$y = 1,\, \frac{1}{2},\, \frac{1}{3},\, \frac{1}{4},\, \frac{1}{5},\, ...\frac{1}{1000}....$$

In terms of the graph, this behavior translates to the graph getting very close, arbitrarily close, to the x-axis. We say that the x-axis, or equivalently the horizontal line $y = 0$, is a **horizontal asymptote** of the function $y = \dfrac{1}{x}$.

Observe that as x is getting close to 0 from the right, the values are becoming very large. Let's test a few positive inputs close to 0:

$$x = \frac{1}{2},\, \frac{1}{3},\, \frac{1}{4},\, \frac{1}{5},\, ...\frac{1}{1000}...$$

The corresponding values $y = \dfrac{1}{x}$ are:

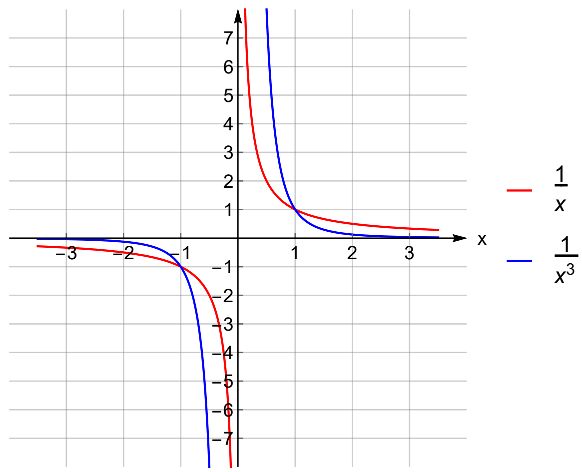
$$y = 2,\, 3,\, 4,\, 5,\, ...1000\,...$$

In terms on the graph, this behavior translates to the graph getting very close to the y-axis with the values of the function becoming arbitrarily large. We say that the y-axis, or equivalently the vertical line $x = 0$, is a **vertical asymptote** of the function $y = \dfrac{1}{x}$.

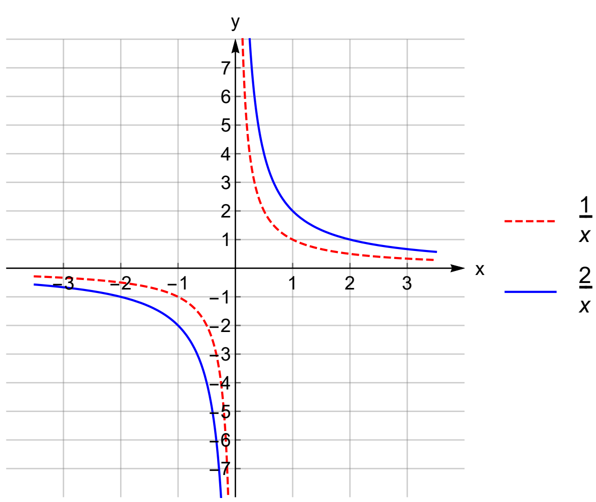
The portion of the graph corresponding to negative inputs x, is symmetric about the origin to the portion of the graph corresponding to positive inputs. Indeed, when x changes sign $y = \frac{1}{x}$ changes sign:

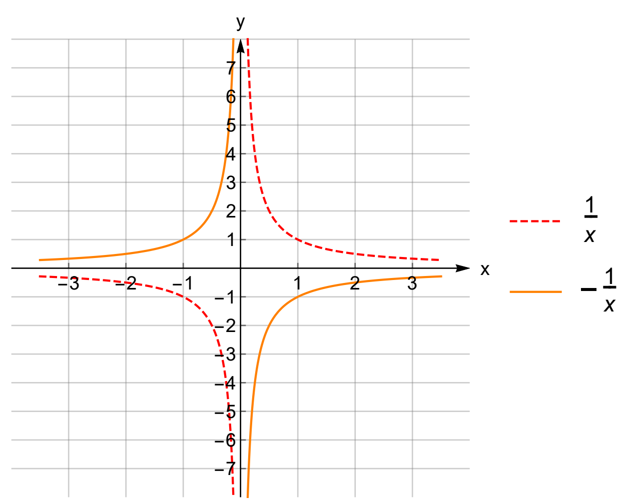
$$\frac{1}{-x} = -\frac{1}{x}.$$

Graphs of other power functions $y = kx^p$ in which the exponent p is an odd negative integer have a similar shape to the graph of $y = \dfrac{1}{x}$. For example:



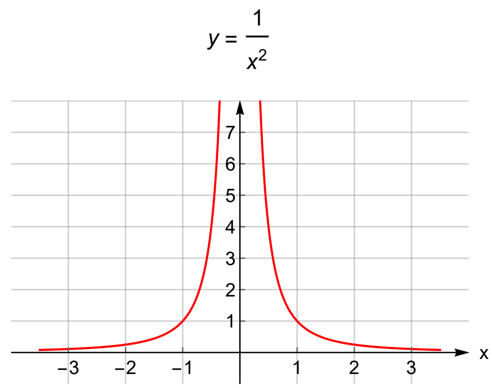
As always, the coefficient k in $y = kx^p$ stretches or shrinks the graph of $y = x^p$ vertically. Additionally, if k is negative, the graph is reflected about the x-axis:



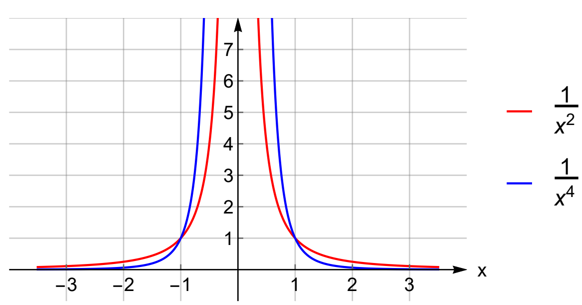


Even Negative Exponents

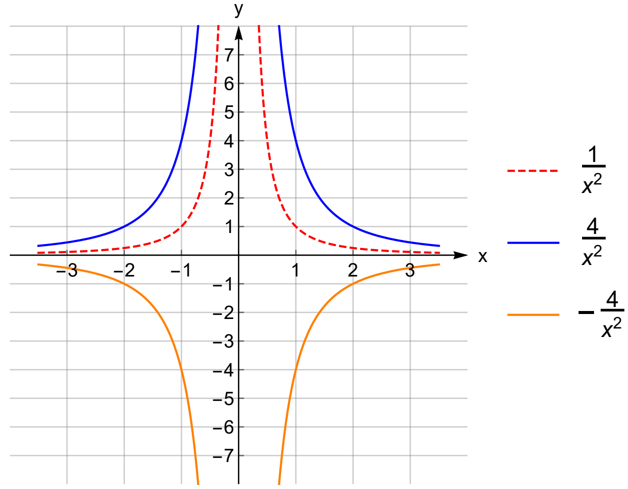
When p is a negative even integer, the graph of $y = x^p$ is entirely above the x-axis and it is symmetric about the y-axis as for p even we have $(-x)^p = x^p$. For example:



The graphs of the functions $y = x^p$ for other even negative integers p have a similar shape:

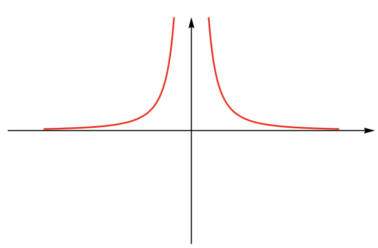
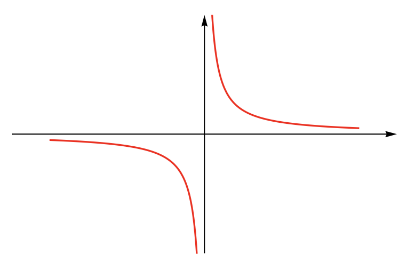
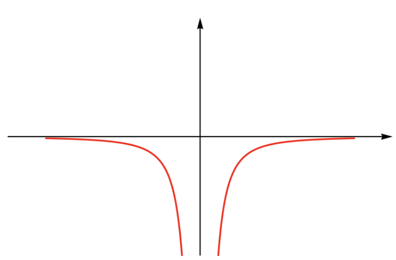
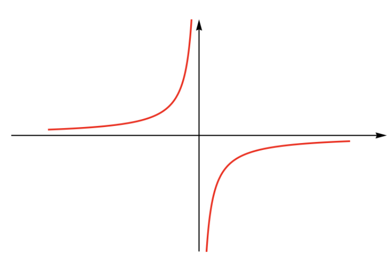


The x-axis is still a horizontal asymptote and the y-axis is a vertical asymptote. The coefficient k is responsible for vertical scaling and a flip about the x-axis if k is negative. For example:



Summary

Here is a summary of how graphs of power functions look for negative integer exponents p, even and odd, and for coefficients k positive and negative:

$f(x) = kx^p$ p negative integer	Even Exponent: p even	Odd Exponent: p odd
Positive Coefficient $k > 0$		
Negative Coefficient $k < 0$		

Fractional Exponents

Example 4.4.3. Which of the functions below are power functions? Those which are, rewrite in standard form $y = kx^p$. Identify the coefficient and the exponent.

(a) $f(x) = \dfrac{3x^3}{7\sqrt{x}}$ (b) $g(x) = \dfrac{2x^{1/3}}{x^4}$ (c) $h(x) = \sqrt{\dfrac{2x}{x^{-2}}}$.

Solution. (a) The function $f(x)$ is a power function. Using rules of exponents for arbitrary exponents, we can rewrite:

$$f(x) = \frac{3x^3}{7\sqrt{x}} = \frac{3x^3}{7x^{1/2}} = \frac{3x^{5/2}}{7} = \frac{3}{7}x^{\frac{5}{2}}$$

The coefficient $k = \frac{3}{7}$, the exponent $p = \frac{5}{2}$.

(b) $g(x)$ is a power function:

$$g(x) = \frac{2x^{1/3}}{x^4} = 2x^{\frac{1}{3}-4} = 2x^{-\frac{11}{3}}$$

The coefficient $k = 2$, the exponent $p = -\frac{11}{3}$.

(c) $h(x)$ is a power function as well. Using rules of exponents for arbitrary exponents and properties of the square root, we can rewrite:

$$h(x) = \sqrt{\frac{2x}{x^{-2}}} = \frac{\sqrt{2x}}{\sqrt{x^{-2}}} = \frac{\sqrt{2} \cdot x^{1/2}}{(x^{-2})^{1/2}} = \frac{\sqrt{2} \cdot x^{1/2}}{x^{-1}} = \sqrt{2} \cdot x^{(1/2-(-1))} = \sqrt{2} \cdot x^{\frac{3}{2}}$$

The coefficient $k = \sqrt{2}$, the exponent $p = \frac{3}{2}$.

Example 4.4.4. Body surface area (BSA) is the total surface area of the human body. The body surface area is used in many measurements in medicine, including the calculation of drug dosages and the amount of fluids to be administered intravenously. There are several accepted formulas to calculate BSA[4,5]. One of the most commonly used is the Du Bois formula:

$$\text{BSA} = 0.007184 \cdot w^{0.425} \cdot h^{0.725}$$

where w is weight in kilograms (kg) and h is height in centimeters (cm). The formula gives BSA in square meters, (m^2).

(a) Calculate BSA of a female who stands 158 cm tall and weighs 60 kg. Give units with your answer.

(b) Let weight be fixed at 70 kg. Then BSA depends on height h only— with weight fixed at 70, BSA is a function of h only. Denote this function by $S(h)$. Find a formula for $S(h)$. Is it a power

[4] https://en.wikipedia.org/wiki/Body_surface_area, accessed: 7/3/20
[5] https://www.medicinenet.com/script/main/art.asp?articlekey=39851, accessed: 7/3/20

function? If yes, identify the coefficient and the exponent. (Give 6 decimal places for your coefficient.)

(c) Use your formula for $S(h)$ to calculate BSA for a female that weighs 70 kg and is 158 cm tall. Give units with your answer.

Solution.

(a) We want to calculate BSA for $w = 60$ and $h = 158$. We substitute the values into the Du Bois formula:

$$\text{BSA} = 0.007184 \cdot 60^{0.425} \cdot 158^{0.725} \approx 1.61.$$

The total body surface area of a person who weighs 60 kg and is 158 cm tall is 1.61 m^2.

(b) Fix $w = 70$. Then BSA as a function of h is:

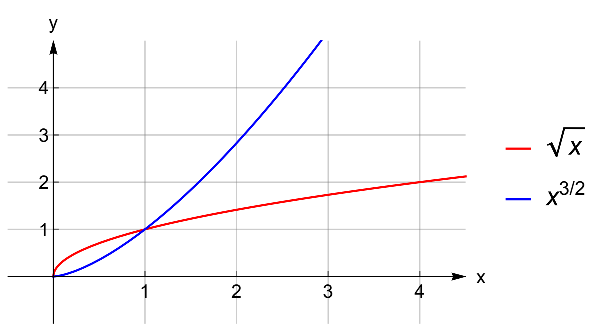
$$S(h) = 0.007184 \cdot 70^{0.425} \cdot h^{0.725} = 0.043705 \cdot h^{0.725}$$

$S(h)$ is a power function of h with the coefficient 0.043705 and exponent 0.725.

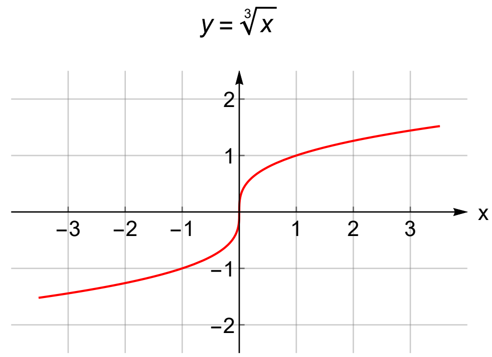
(c) BSA of a female who weighs 70 kg and is 158 cm tall is $S(158) = 0.043705 \cdot 158^{0.725} \approx 1.72$. The total body surface area of a person who weighs 60 kg and is 158 cm tall is 1.72 m^2.

Graphs of Power Functions: Fractional Exponents

When considering power functions with fractional exponents or, more general, with exponents which are not integers, we restrict the domain to $x \geq 0$ (or to $x > 0$ if a non-integral exponent is negative). The graphs of $x^{1/2}$ and $x^{3/2}$ are given below as an example:



Traditionally, roots of odd orders like $\sqrt[3]{x}$ or $\sqrt[5]{x}$ are considered and graphed for all x:



Practice Problems for Section 4.4

In Problems 1-6, determine whether a given function is a power function. If yes, identify the coefficient k and the exponent p. If not, say so.

1. $y = \dfrac{\sqrt{x^3}}{0.1x}$

2. $y = \dfrac{\sqrt{0.01x^3}}{\sqrt{x}}$

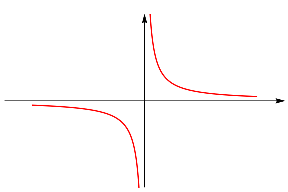
3. $g(x) = \dfrac{x^{\frac{5}{3}}}{4x^{\frac{1}{4}}}$

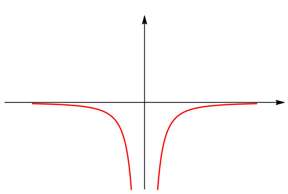
4. $h(x) = 2x^{\frac{3}{2}} - x^{-2}$

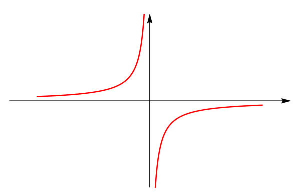
5. $y = \sqrt{\dfrac{x^5}{9x}}$

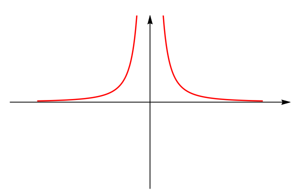
6. $f(x) = (\sqrt{2^x})^2$

In Problems 7-10 you see the graph of a power function $y = kx^p$ where p is a negative integer. In each of the graphs, is the exponent p even or odd? Is the coefficient k positive or negative?

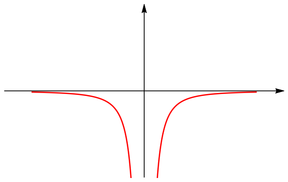
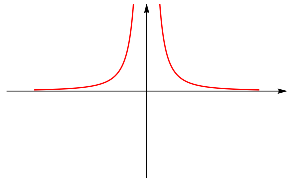
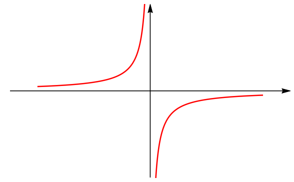
7. 

8. 

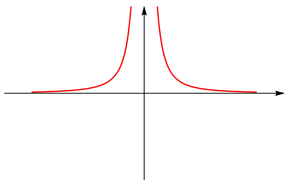
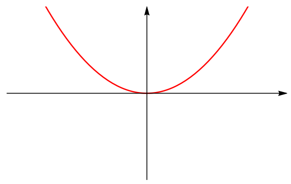
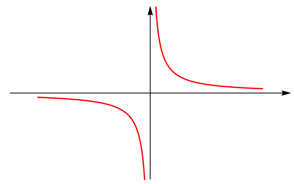
9. 

10. 

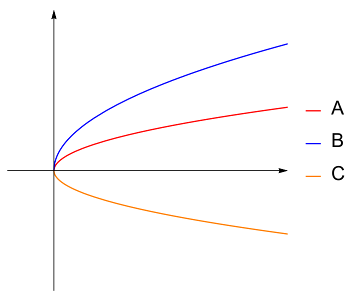
11. Which of the graphs below is the graph of the function $f(x) = -2x^{-5}$?

(A)  (B)  (C) 

12. Which of the graphs below is the graph of the function $f(x) = 2x^{-4}$?

(A)  (B)  (C) 

13. Below you see graphs of the functions $y = x^{\frac{1}{2}}$, $y = 2x^{\frac{1}{2}}$, and $y = -x^{\frac{1}{2}}$. Decide which is which.



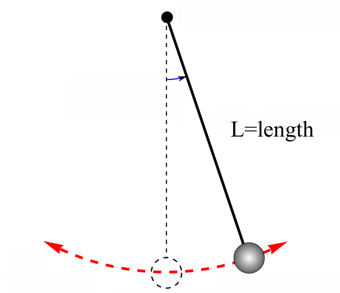
14. Find a formula for a power function $f(x) = kx^p$ given numerically by:

x	0	1	2	3	5
$f(x)$	0	3	$3\sqrt{2}$	$3\sqrt{3}$	$3\sqrt{5}$

15. Find a formula for a power function $g(x) = kx^p$ given numerically by:

x	0	1	2	3	5
$g(x)$	undefined	2	1	$\frac{2}{3}$	$\frac{2}{5}$

16. Consider a pendulum depicted below[6]:



When the pendulum is displaced sideways from its resting position — called the equilibrium position — the force due to gravity will cause the pendulum to oscillate back and forth about the equilibrium position. The time, T, needed to execute one full cycle — a left swing and the right swing — is called the pendulum's period. The period T depends on the length L of the pendulum[7] and the local acceleration due to gravity g:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

17. With g fixed, is T a power function of L? In other words, is $T = kL^p$? If yes, find the coefficient and the exponent.

18. On the surface of the Earth, the acceleration due to gravity, g, is equal to 9.81 m/sec^2, where m stands for meters. Calculate the period of a pendulum of length 0.5 m that happily oscillates in Kingston, RI.

19. An astronaut is standing on the surface of a faraway asteroid wondering about the acceleration due to gravity on the asteroid. The astronaut has a watch and a pendulum whose length is 0.3 m. The astronaut measures the period of the pendulum which turns out to be 5 seconds. What is the acceleration due to gravity on the asteroid?

[6] Modified from a public domain image at https://en.wikipedia.org/wiki/Pendulum, accessed: 7/8/20

[7] https://en.wikipedia.org/wiki/Pendulum, accessed: 7/8/20

Chapter 5

Exponential Functions

5.1 Exponential Functions: Practical Meaning

Linear functions $y = f(t) = mt + b$ which you studied in Chapter 2 change by a constant **amount**, m, per unit change in t. Exponential functions change by a constant **factor**.

Example 5.1.1. You deposit \$1,500 into a savings account that pays 5% interest annually. Let $B = B(t)$ be your balance after t years.

(a) What is $B(0)$, $B(1)$, $B(2)$, and $B(3)$?

(b) Find a formula for the function $B(t)$.

(c) What is your balance after 10 years?

Solution. (a) For $t = 0$, $B(0)$ is equal to your initial deposit; that is, $B(0) = 1500$.

After 1 year, the bank will add to your balance the interest of 5% of your initial amount $B(0)$. 5% of $B(0)$ is $0.05 \cdot B(0)$. So your balance after 1 year is:

$$B(1) = B(0) + 0.05 \cdot B(0).$$

Equivalently:

$$B(1) = B(0)(1 + 0.05) = B(0) \cdot 1.05.$$

Adding 5% of a quantity to it means multiplying the quantity by the factor $(1 + 0.05) = 1.05$. Since $B(0) = 1500$, we have:

$$B(1) = B(0) \cdot 1.05 = 1500 \cdot 1.05 = 1575.$$

During the first year, your money earned \$75 which is 5% of 1500.

At the end of the second year, the bank will add to your account 5% of 1575 - the interest for the first year is earning interest during the second year. That is, the bank will add 5% of $B(1) = 1575$ to it. Or, equivalently, the bank will multiply $B(1)$ by 1.05. Hence:

$$B(2) = B(1) \cdot 1.05 = 1575 \cdot 1.05 = 1653.75.$$

Notice that $1,653.75 - 1575 = 78.75$. So during the second year your money made \$78.75 which is 5% of 1575.

The amount of money by which your balance is increasing each year is not constant. But the factor by which your balance is increasing each year is constant and equal to 1.05. Also, the percentage by which your balance is increasing each year is constant and equal 5%. To see a clear pattern, observe that since $B(1) = B(0) \cdot 1.05$:

$$B(2) = B(1) \cdot 1.05 = (B(0) \cdot 1.05) \cdot 1.05 = B(0) \cdot 1.05^2.$$

Similarly:

$$B(3) = B(2) \cdot 1.05 = (B(0) \cdot 1.05^2) \cdot 1.05 = B(0) \cdot 1.05^3.$$

Substituting $B(0) = 1500$, we get $B(3) = 1500 \cdot 1.05^3 = 1736.44$. Your balance after 3 years is \$1,736.44.

(b) Each year your current balance is multiplied by the factor 1.05. So after t years, your initial balance $B(0)$ will be multiplied t times by 1.05:

$$B(t) = B(0) \cdot 1.05^t = 1500 \cdot 1.05^t.$$

So $B(t) = 1500 \cdot 1.05^t$ gives your balance after t years.

(c) Using the latter formula, we calculate $B(10) = 1500 \cdot 1.05^{10} = 2443.34$. Your balance after 10 years is \$2,443.34.

A function of the form $B(t) = 1500 \cdot 1.05^t$ is called an exponential function as the independent variable t is in the exponent. In general:

Definition of an Exponential Function

Let A and b be given constants such that $A > 0$, $b > 0$, and $b \neq 1$. Then the function:

$$f(t) = A \cdot b^t$$

is called an **exponential function** with the base b and initial value A. b is also called the **growth factor**.

The function $B(t) = 1500 \cdot 1.05^t$ of Example 5.1.1 is an exponential function with the base, or equivalently with the growth factor, $b = 1.05$ and the initial value $A = 1500$.

Notice that for any exponential function $f(t) = A \cdot b^t$:

$$f(0) = A \cdot b^0 = A$$

as $b^0 = 1$ for any b. Hence, A is always the value of an exponential function at 0, hence the name "initial value".

Exponential functions are used very often to model real-life processes of exponential growth and exponential decay.

Exponential Growth

If the base, or equivalently the growth factor, b, is greater than 1, the exponential function $f(t) = A \cdot b^t$ is increasing. Indeed, for every unit change in t, we multiply the current value of the function by the factor b which is greater than 1. So we increase the value. Hence, as t increases, $f(t)$ increases.

The function $B(t) = 1500 \cdot 1.05^t$ in Example 5.1.1 is increasing as its base $b = 1.05$ is greater than 1. The growth of the amount of money in your savings account is exponential and is given by an exponential function $B(t) = 1500 \cdot 1.05^t$.

Whenever the base b is greater than 1, b, for some positive number r, can be written as:

$$b = 1 + r.$$

The constant r written in the percentage form is called the **percent growth rate**. For $B(t) = 1500 \cdot 1.05^t$, $1.05 = 1 + 0.05$. So $r = 0.05$. In other words, $r = 5\%$. The annual percent growth rate of your money is 5%. The percent growth rate $r = 5\%$ is the percentage of the current balance that is added to your account every year.

Exponential functions are often used to model population growth for all kinds of populations: human populations, animal populations or populations of bacteria or insects in laboratory experiments.

Bacteria, which are one-cell organisms, reproduce by each cell dividing into two daughter cells with a frequency that depends on the kind of bacteria. Such a population grows slowly at first when the population consist of a small number of bacteria, and faster and faster with a larger and larger number of bacteria ready to divide. It is reasonable to expect that the population will grow not by a constant number of bacteria but by a constant percentage of its current size. In other words, we expect the population to grow exponentially.

Example 5.1.2. A population of Escherichia coli (E. coli) bacteria in nutrient-rich laboratory conditions grows by 3.53 percent per minute[1]. Let $P = f(t)$ be the number of bacteria t minutes after the experiment began. Initially, at $t = 0$, we have 2000 bacteria.

(a) What is the percent growth rate? What is the growth factor? Write a formula for $f(t)$.

[1]https://www.ncbi.nlm.nih.gov/pmc/articles/PMC6015860/, accessed: 6/26/20

(b) How many bacteria are there at $t = 20$? At $t = 40$? At $t = 60$?

Solution. (a) Since the population increases by a constant percent, $f(t)$ is an exponential function, $f(t) = A \cdot b^t$. The initial amount is 2000, so $A = 2000$. Every minute the population increases by 3.53% of its current size, so the percent growth rate is $r = 3.53\%$ or equivalently $r = 0.0353$. The growth factor is $b = 1 + r = 1 + 0.0353 = 1.0353$. Hence:

$$P = f(t) = 2000 \cdot 1.0353^t.$$

(b) We use the formula obtained is (a) and calculate: $f(20) = 2000 \cdot 1.0353^{20} = 4002.7$, $f(40) = 2000 \cdot 1.0353^{40} = 8010.8$, and $f(60) = 2000 \cdot 1.0353^{60} = 16032.6$.

You may notice that the E. coli population approximately doubles during the first 20 minutes, then doubles again during the next 20 minutes, and again after 20 more minutes. This is not a coincidence. Processes of exponential growth have the so-called doubling time; that is, the time needed for the quantity to double. We will study it closely in Section 5.3.

Exponential Decay

If the base b of an exponential function $f(t) = A \cdot b^t$ is less than 1, $b < 1$, multiplying by b decreases the value. So the exponential function is decreasing. Commonly, b is called the "growth factor" in both cases, $b > 1$ and $b < 1$, even though for $b < 1$ we have a decay rather than growth.

The base b can still be written as:

$$b = 1 + r$$

for some number r. Except that for b less than 1, r is negative. Again, r is called the "percent growth rate". This terminology is counterintuitive although one can argue that a negative growth rate means decay.

Many real-life processes are modeled by decaying exponential functions. For example, elimination of a drug from the body, processes of radioactive decay, and many others.

Example 5.1.3. A common anti-anxiety medication Diazepam is eliminated from the body very slowly at the rate of 26% daily. Suppose that a patient takes a one-time dose of 10 mg of Diazepam. Let $D = D(t)$ be the amount of Diazepam left in his bloodstream t days after the dose. Find a formula for $D(t)$.

The initial amount, $D(0) = 10$. During the first day, 26% of the initial dose is eliminated. Hence:

$$D(1) = D(0) - 0.26 \cdot D(0) = D(0)(1 - 0.26) = D(0)(1 + (-0.26)).$$

During the second day, 26% of $D(1)$ is eliminated:

$$D(2) = D(1) - 0.26 \cdot D(1) = D(1)(1 + (-0.26)) = D(0)(1 + (-0.26))^2$$

and so on. Each next day we multiply the dose from the day before by $(1 + (-0.26))$. The amount left after t days is:

$$D(t) = D(0)(1 + (-0.26))^t.$$

Hence, $D(t)$ is an exponential function. The growth factor $b = (1 + (-0.26)) = 0.74$, the daily percent growth rate is $r = -0.26$, and the initial amount $A = D(0) = 10$. The final version of the formula is:

$$D(t) = 10 \cdot 0.74^t.$$

Of course, every day 26% of the amount from the day before gets eliminated, so 74% of the amount from the day before stays in the body.

The growth factor 0.74 is less than 1 and the daily percent growth rate of -26% is negative since the amount is decaying. Often we say that the daily **percent decay rate** is 26%.

Notice that after 2 days, the patient still has $D(2) = 10 \cdot 0.74^2 = 5.48$ mg in his system — more than a half of the initial dose. Such a slow elimination rate causes a medication buildup if a daily dose is taken.

For the sake of convenience, let's summarize the relationship between the growth factor b and the percent growth rate r.

General Rules:

Let an exponential function $f(t) = A \cdot b^t$ be given (increasing or decreasing). Then:

- b is called the "growth factor".
- The constant r such that $b = 1 + r$ is called the "percent growth rate" or simply the "growth rate".
- Given r, we calculate b by taking $b = 1 + r$.
- Given b, we calculate r by taking $r = b - 1$.

Another common real-life application of exponential functions is in the study of radioactivity. All radioactive isotopes decay and they decay exponentially. Some extremely slowly, some very fast. Carbon-14, a radioactive isotope of Carbon used in carbon dating, decays so slowly that it takes thousands of years for half of an initial amount to decay. Iodine-131 used in the treatment of thyroid cancer, decays in a matter of days.

Example 5.1.4. The daily percent decay rate of Iodine-131 is 8.3%. Let $I = I(t)$ be the amount of Iodine-131, in μg, left after t days if the initial amount is 10μg.

(a) Find a formula for $I(t)$. What is the growth factor? What is the percent growth rate?

(b) Estimate how long it will take for half of initial amount to decay.

Solution. (a) The daily percent decay rate is 8.3%. Hence, $r = -0.083$ and the growth factor is $b = (1 + (-0.083)) = 0.917$. The initial amount is 10μg. Hence:

$$I(t) = 10 \cdot 0.917^t.$$

(b) We are looking for t such that $I(t) = 5$. At this point, we don't have tools other than trial and error. Let's calculate a few values: $I(10) = 10 \cdot 0.917^{10} = 4.2$. 10 days is a bit too long. Try $I(7) = 10 \cdot 0.917^7 = 5.45$. 7 days is too short. Try $I(8) = 10 \cdot 0.917^8 = 4.9998$. 8 days is close enough. It will take about 8 days for half of the initial amount to decay.

Note: From now on, we will often use the more common notation: $I(t) = 10(0.917)^t$ as well as the notation $I(t) = 10 \cdot 0.917^t$.

Example 5.1.5. The following exponential functions describe populations of four towns t years after January 1, 2000:

(a) $P(t) = 700(1.15)^t$ (b) $A(t) = 10000(0.89)^t$

(c) $Q(t) = 12000(0.92)^t$ (d) $S(t) = 1500(1.021)^t$.

For each town, identify the initial population at $t = 0$, the growth factor, and the annual percent growth rate. For each town determine if its population is increasing or decreasing.

Solution. (a) The initial population is 700 people as $P(0) = 700(1.15)^0 = 700$. The growth factor is 1.15. The percent growth rate $r = 1.15 - 1 = 0.15 = 15\%$. Since the growth factor of 1.15 is greater than 1, the population of the town is increasing.

(b) The initial population is 10000 people. The growth factor is 0.89. The percent growth rate $r = 0.89 - 1 = -0.11 = -11\%$. In other words, the decay rate is 11%. The growth factor of 0.89 is less than 1 so the population of the town is decreasing.

(c) The initial population is 12000 people. The growth factor is 0.92. The percent growth rate $r = 0.92 - 1 = -0.08 = -8\%$. In other words, the decay rate is 8%. The population of the town is decreasing.

(d) The initial population is 1500 people. The growth factor is 1.021. As $1.021 = 1 + 0.021$, the percent growth rate is 2.1%. The growth factor of 1.021 is greater than 1 so the population of the town is increasing.

Example 5.1.6. Let $V(t)$ be the value, in dollars, of an antique lamp t years after its purchase. The lamp was purchased for \$5000 and its value $V(t)$ increases by \$400 each year.

(a) Find a formula for the function $V(t)$. What kind of function is it?

(b) What is the rate of increase of $V(t)$ in dollars per year?

(c) What is the value of the lamp 10 years after its purchase?

Solution. (a) The value $V(t)$ increases by a fixed amount of dollars each year. Hence, $V(t)$ is a linear function:

$$V(t) = 5000 + 400t$$

(b) The slope $m = 400$ represents the constant rate of increase of the value $V(t)$ in dollars per year.

(c) The value after 10 years is $V(10) = 5000 + 400 \cdot 10 = 9000$ dollars.

Example 5.1.7. Let $V(t)$ be the value, in dollars, of an antique lamp, t years after its purchase. The lamp was purchased for \$5000 and its value $V(t)$ increases by 12% each year.

(a) Find a formula for the function $V(t)$. What kind of function is it?

(b) What is the growth factor and the annual percent growth rate of $V(t)$?

(c) What is the value of the lamp 10 years after its purchase?

Solution. (a) This time the value of the lamp is increasing not by a constant amount each year but by a constant percentage of the current value each year. An increase of 12% means adding 12% of $V(t)$ to itself; that is, multiplying $V(t)$ by a constant factor 1.12 each year. Hence, the function $V(t)$ is exponential and equal to:

$$V(t) = 5000(1.12)^t$$

(b) The growth factor is 1.12; the annual percent growth rate is 0.12=12%.

(c) The value after 10 years is $V(10) = 5000(1.12)^{10} = 15529.24$ dollars.

Example 5.1.8. Is a given function $Q(t)$ exponential? If yes, rewrite $Q(t)$ in the form $Q(t) = A \cdot b^t$. Identify A and b.

(a) $Q(t) = 50(2)^{\frac{t}{6}}$

(b) $Q(t) = 550(t^2)^{\frac{1}{3}}$

(c) $Q(t) = (\sqrt{2})^{2t}$

(d) $Q(t) = 12000(\frac{1}{8})^{\frac{t}{3}}$.

Solution. (a) We have $Q(t) = 50(2)^{\frac{t}{6}} = 50((2^{\frac{1}{6}})^t) = 50(\sqrt[6]{2})^t)$. Hence, $Q(t)$ is exponential with $A = 50$ and $b = \sqrt[6]{2}$. Since $\sqrt[6]{2} \approx 1.1225$, in an applied problem we would write $Q(t) = 50(1.1225)^t$.

(b) We can simplify $Q(t)$ as $Q(t) = 550t^{\frac{2}{3}}$. $Q(t)$ is a power function but not an exponential function. Note that in exponential functions the independent variable appears in the exponent and the base is constant. It is the other way around in power functions: the base changes but the exponent is constant.

(c) $Q(t) = ((\sqrt{2})^2)^t = 2^t$. The function $Q(t)$ is exponential with $A = 1$ and $b = 2$.

(d) Simplifying we get: $Q(t) = 12000((\frac{1}{8})^{1/3})^t = 12000(\frac{1}{2})^t$. The function $Q(t)$ is exponential with $A = 12000$ and $b = \dfrac{1}{2}$.

Practice Problems for Section 5.1

In Problems 1-4, decide if a given exponential function represents a process of exponential growth or decay. For each function identify the initial value and the growth factor.

1. $f(t) = 120(1.7)^t$
2. $g(t) = 1500(0.95)^t$
3. $h(t) = 60(0.83)^t$
4. $m(t) = 130(1.02)^t$

5. You deposit \$2,000 dollars into a savings account that pays 3.5% annually. Let $B = B(t)$ be your balance t years later.

   (a) Find a formula for $B(t)$ in the form $B(t) = A \cdot b^t$.

   (b) Find your balance after 7 years.

   (c) Find the growth factor and the percent growth rate of the function $B(t)$.

6. Let $V(t)$ be the value, in dollars, of an antique desk t years after its purchase. The desk was purchased for \$7000 and its value $V(t)$ increases by \$500 per year.

   (a) Find a formula for the function $V(t)$. What kind of function is it?

   (b) What is the rate of increase of $V(t)$ in dollars per year?

   (c) What is the value of the desk after 12 years?

7. Let $V(t)$ be the value, in dollars, of an antique desk, t years after its purchase. The desk was purchased for \$7000 and its value $V(t)$ increases by 9.5% per year.

   (a) Find a formula for the function $V(t)$. What kind of function is it?

   (b) What is the growth factor and the percent growth rate of $V(t)$?

   (c) What is the value of the desk after 12 years?

8. Let $P(t)$ be the population of a town t years after the year 1990. The population $P(t)$ was 12000 people in 1990, that is, at $t = 0$, and it has been increasing by 750 people each year.

(a) Write a formula for the function $P(t)$. What kind of function is it?

(b) What is the rate of increase of the population in people per year?

9. Let $P(t)$ be the population of a town t years after the year 1990. The population $P(t)$ was 12000 people in 1990, that is, at $t = 0$, and it has been increasing by the factor 1.107 each year.

   (a) Write a formula for the function $P(t)$. What kind of function is it?

   (b) What is the growth factor and the annual percent growth rate of the population?

10. A biologist studies the effects of three different nutrients, a, b, and c, on the growth of a particular kind of bacterium. t hours after the experiment began, the number of bacteria in the culture fed nutrient a is $A(t)$, the number of bacteria in the culture fed nutrient b is $B(t)$, the number of bacteria in the culture fed nutrient c is $C(t)$. The biologist observes that the functions $A(t)$, $B(t)$ and $C(t)$ are given by the following formulas:

    $A(t) = 500(1.09)^t, \qquad B(t) = 500(1.15)^t, \qquad C(t) = 500(0.75)^t.$

    (a) Which of the nutrients stimulates growth of the bacteria the most? What is the hourly percent growth rate of the culture fed that nutrient?

    (b) One of the nutrients proves toxic to the bacterium. Which one is it? What is the percent growth rate of the culture fed that nutrient?

11. Following a dose of 40 mg, a medication leaves a patient's body at an hourly percent rate of 3.7%.

    (a) Write a formula for the amount $M(t)$ of the medication, in mg, left in the body t hours after the dose. What is the growth factor of $M(t)$?

    (b) How much of the medication remains in the body after 10 hours?

12. Cesium-137, a radioactive isotope of Cesium, decays very slowly. Let 25 mg of Cesium-137 be present initially and let $C(t)$ be the amount, in mg, remaining after t years. Then:

    $$C(t) = 25(0.9772)^t$$

    (a) What is the growth factor of Cesium-137?

    (b) What is the percent growth rate?

    (c) How much Cesium-137 remains after 30 years?

In Problems 13-18, decide whether a given function is exponential. If yes, rewrite the function in the form $y = A \cdot b^t$. Identify the initial value, the growth factor, and decide if the function is increasing or decreasing. Round off your answers to four decimal places.

13. $y = 30(0.8)^{2t}$

14. $y = \sqrt{160(1.09)^{2t}}$

15. $y = 30\sqrt{t^4}$

16. $y = 70(2^{\frac{t}{5}})$

17. $y = 5(2^{t-1})$

18. $y = 60\left(\frac{1}{2}\right)^{\frac{t}{3}}$

5.2 Graphs of Exponential Functions

In the previous section we looked at algebraic properties of exponential functions and applied examples involving exponential functions. In this section we study graphs of exponential functions.

The graph of an exponential function $f(t) = A \cdot b^t$ depends on whether the growth factor b is greater than 1 or less than 1. Of course: if $b > 1$, the function $f(t) = A \cdot b^t$ is increasing — its graph is climbing. If $b < 1$, the function $f(t) = A \cdot b^t$ is decreasing — its graph is falling.

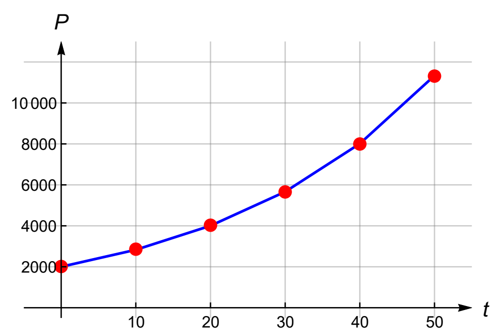
Example 5.2.1. Let's graph the function of Example 5.1.2 that describes the population P of E. coli bacteria in a laboratory culture t minutes after the experiment began:

$$P = f(t) = 2000(1.0353)^t.$$

The table of values for t taken every 10 minutes is as follows:

t (minutes)	0	10	20	30	40	50
P (# of bacteria)	2000	2829.4	4002.7	5662.6	8010.8	11332.9

The graph reflecting the data is:



The function increases and moreover it increases faster and faster; the population of E. coli bacteria increases faster and faster.

Observe that we drew the graph only for $t \geq 0$ as the experiment begins at $t = 0$. Mathematically, the function $P = 2000(1.0353)^t$ is defined for all t.

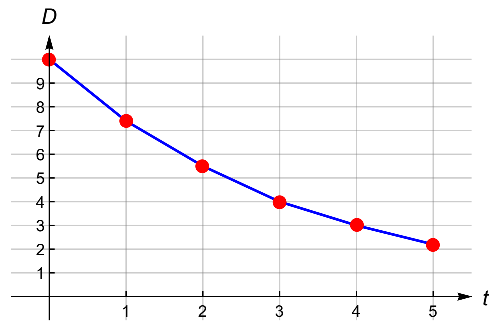
Example 5.2.2. Let's look at the function

$$D(t) = 10(0.74)^t$$

that shows the amount of Diazepam, in mg, left in the body t days after a single 10 mg dose. Since the growth factor 0.74 is less than 1, $D(t)$ is a decreasing function. Here are the values of $D(t)$ for $t = 0, 1, 2, 3, 4, 5$:

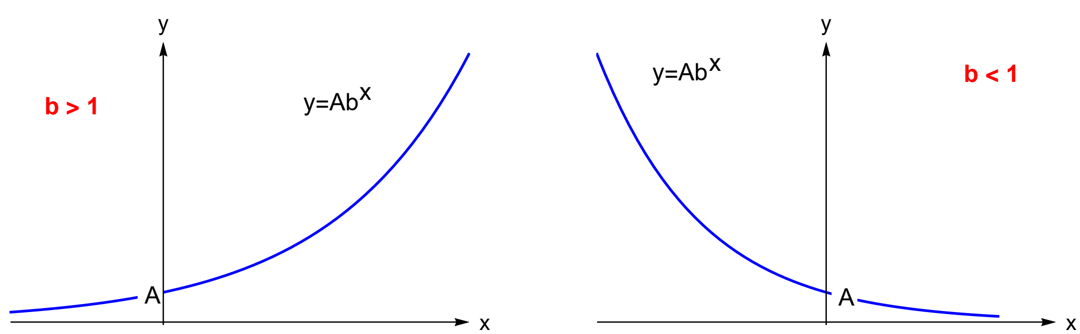
t (days)	0	1	2	3	4	5
D (mg)	10	7.4	5.5	4.0	3.0	2.2

Here is the corresponding graph:



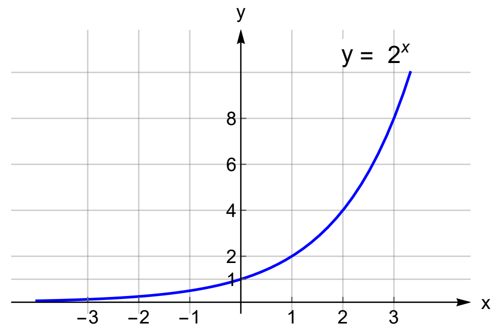
The function is decreasing and it is decreasing slower and slower; that is, the amount by which the function is decreasing daily is less and less as time goes on. Of course, the function decreases daily by 26% of the current amount. The smaller the current amount the smaller the decrease.

In general, the shape of the graph of an exponential function $y = A \cdot b^x$ is different for the base $b > 1$ and for $b < 1$:



When $b > 1$, the function $y = A \cdot b^x$ is increasing so its graph is climbing. For $b < 1$, the graph is falling. In both cases the values of $y = A \cdot b^x$ are always positive so the graph is entirely above the x-axis. The y-intercept is always equal to the initial value A.

Example 5.2.3. Graph $y = 2^x$. We have $A = 1$, $b = 2$. The base is greater than 1 so we expect the graph to be climbing. We create a table of values or use a graphing calculator and obtain:



Note that for $x > 0$, $2^x > 1$ and 2^x increases as x increases:

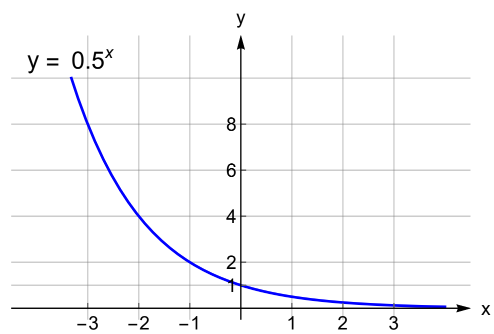
$$2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8 \quad 2^4 = 16 \ ...$$

For $x < 0$, $2^x < 1$ and 2^x becomes smaller and smaller as x becomes "more and more negative":

$$2^{-1} = \frac{1}{2}, \quad 2^{-2} = \frac{1}{2^2} = \frac{1}{4}, \quad 2^{-3} = \frac{1}{2^3} = \frac{1}{8}, \quad 2^{-4} = \frac{1}{2^4} = \frac{1}{16} \ ...$$

In fact, as x becomes "more and more negative", 2^x becomes arbitrarily close to 0 — the graph "snuggles" to the x-axis although it never crosses the x-axis. We say that the graph *asymptotically* approaches the x-axis. The x-axis is a horizontal asymptote of the function $y = 2^x$.

Example 5.2.4. Graph $y = (\frac{1}{2})^x$. We have $A = 1$, $b = \frac{1}{2} = 0.5$. The base is smaller than 1 so we expect the graph to be falling. We create a table of values or use a graphing calculator and obtain:



Note that in this example, $(\frac{1}{2})^x < 1$ as $x > 0$ and $(\frac{1}{2})^x$ decreases as x increases asymptotically approaching the x-axis:

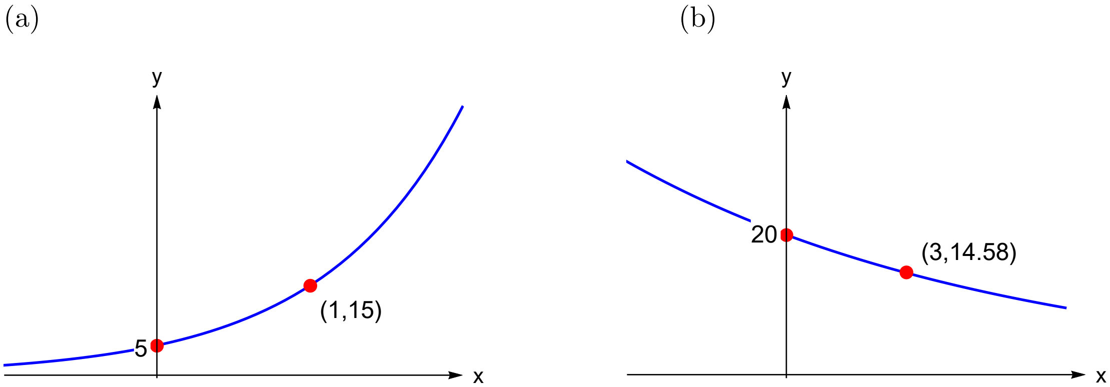
$$\left(\frac{1}{2}\right)^1 = \frac{1}{2}, \quad \left(\frac{1}{2}\right)^2 = \frac{1}{4}, \quad \left(\frac{1}{2}\right)^3 = \frac{1}{8} \ ...$$

The x-axis is a horizontal asymptote of the function $(\frac{1}{2})^x$.

On the other side, as x becomes "more and more negative", $(\frac{1}{2})^x$ becomes larger and larger:

$$\left(\frac{1}{2}\right)^{-1} = \frac{1}{\frac{1}{2}} = 2, \quad \left(\frac{1}{2}\right)^{-2} = \frac{1}{(\frac{1}{2})^2} = \frac{1}{\frac{1}{4}} = 4, \left(\frac{1}{2}\right)^{-3} = \frac{1}{(\frac{1}{2})^3} = \frac{1}{\frac{1}{8}} = 8 \ldots$$

Example 5.2.5. Find a function $f(x) = A \cdot b^x$ with the graph shown below:



Solution. (a) We have to find the initial value A and the growth factor b. Since A is always the y-intercept of $y = A \cdot b^x$, we have $A = 5$ so $f(x) = 5 \cdot b^x$. To find b note that the point $(1, 15)$ is on the graph. Hence, $f(1) = 5 \cdot b^1 = 5 \cdot b = 15$. The equation $5 \cdot b = 15$ gives $b = 3$. Thus, the function corresponding to the graph (a) is $f(x) = 5 \cdot 3^x$.

(b) The initial value A is the vertical intercept. Hence, $A = 20$. We use the point $(3, 14.58)$ to set up an equation for b:

$$\begin{aligned} f(3) &= 14.58 \\ 20 \cdot b^3 &= 14.58 \\ b^3 &= 14.58/20 \\ b &= (0.729)^{1/3} \\ b &= 0.90 \end{aligned}$$

Hence, the function depicted on the graph (b) is $f(x) = 20 \cdot 0.90^x$.

Example 5.2.6. Find a function $f(t) = A \cdot b^t$ whose graph contains points $(1, 15)$ and $(2, 22.5)$.

Solution. $(1, 15)$ is on the graph so $f(1) = 15$. Similarly, $(2, 22.5)$ being on the graph tells us that $f(2) = 22.5$. Hence:

$A \cdot b^1 = 15$ and $A \cdot b^2 = 22.5$.

To find A and b, observe that:

$$\frac{A \cdot b^2}{A \cdot b^1} = \frac{22.5}{15}$$
$$\frac{b^2}{b} = \frac{22.5}{15}$$
$$b = 1.5$$

Hence, $f(t) = A \cdot 1.5^t$. Since $f(1) = 15$, we have $A \cdot 1.5^1 = 15$ which gives $A = \frac{15}{1.5} = 10$. Thus, $f(t) = 10(1.5)^t$.

Practice Problems for Section 5.2

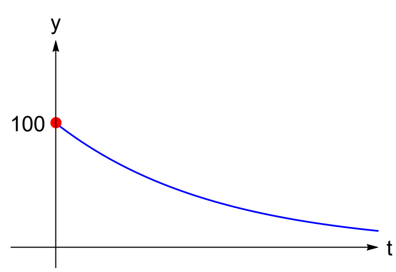
1. Complete the table of values below and use it to graph the exponential function $y = 3^x$ in the interval $-3 \le x \le 3$.

x	-3	-2	-1	0	1	2	3
$y = 3^x$	$\frac{1}{27}$	?	?	?	?	?	?

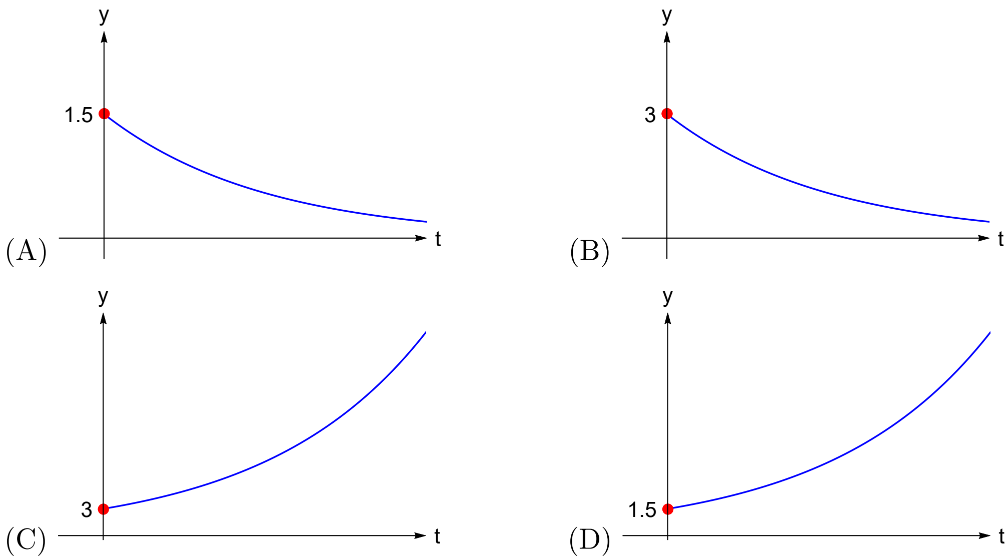
2. Complete the table of values below and use it to graph the exponential function $y = \left(\frac{1}{3}\right)^x$ in the interval $-3 \le x \le 3$.

x	-3	-2	-1	0	1	2	3
$y = \left(\frac{1}{3}\right)^x$	27	?	?	?	?	?	?

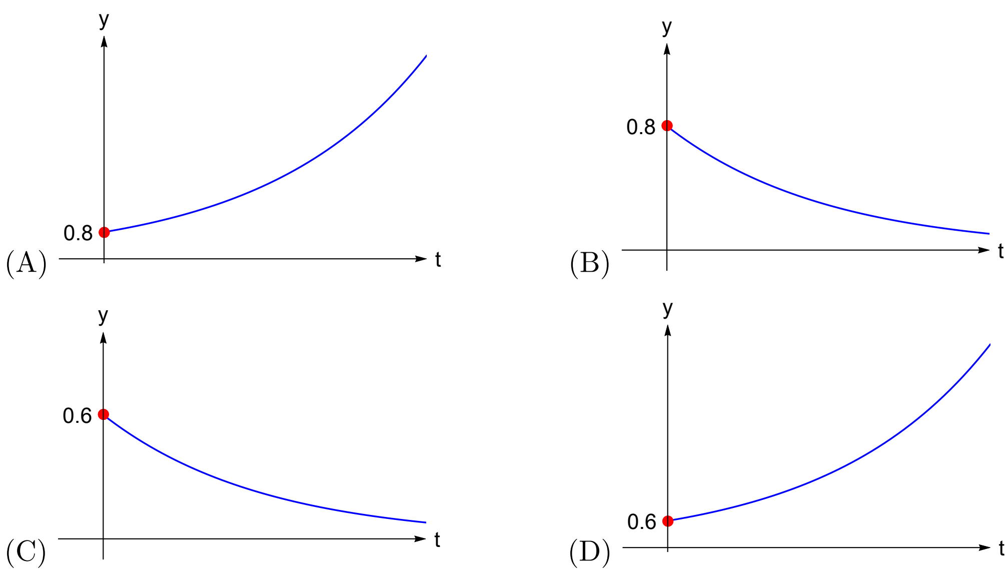
3. Below you see a graph of an exponential function $y = A \cdot b^t$. What is its initial value A? Is its growth factor b greater than 1 or less than 1?



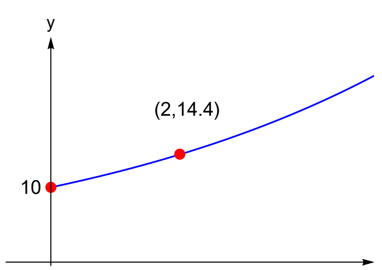
4. Which of the graphs (A)-(D) below could be a graph of the function $y = 3(1.5)^t$? Explain your answer!



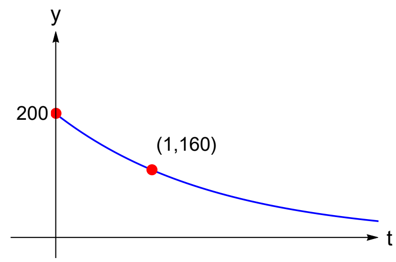
5. Which of the graphs (A)-(D) below could be a graph of the function $y = 0.8(0.75)^t$? Explain your answer!



6. Find a formula in the form $y = A \cdot b^t$ for the exponential function whose graph is given below:



7. Find a formula in the form $y = A \cdot b^t$ for the exponential function whose graph is given below:



8. Find a function $f(x) = A \cdot b^x$ whose graph contains points $(0, 500)$ and $(4, 120.05)$.

9. Find a function $g(x) = A \cdot b^x$ whose graph contains points $(1, 48)$ and $(2, 76.8)$.

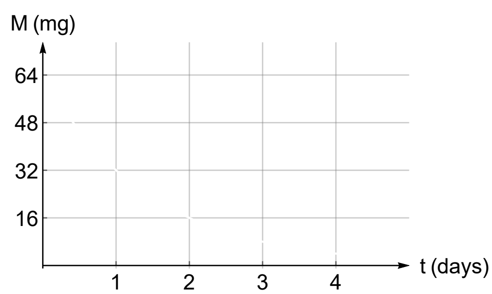
10. A medication is eliminated from the body at the daily percent rate of 50%. A patient takes a single dose of 64 mg of the medication. Let $M(t)$ be the amount, in mg, remaining in the patient's body t days after the dose.

    (a) Find a formula for the function $M(t)$.

    (b) Fill in the missing entries in the following table of values:

t (days)	0	1	2	3	4
$M(t)$ mg	?	?	?	?	?

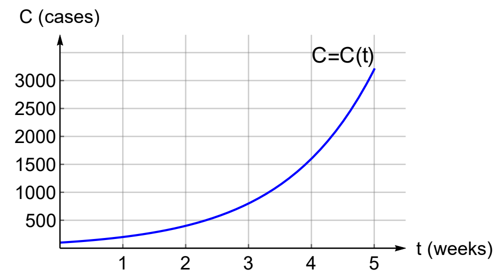
    (c) Use the table to plot the points corresponding to $t = 0, 1, 2, 3, 4$ that are on the graph of $M(t)$. Sketch a graph of the function $M = M(t)$ in the coordinate system below.



11. The number of diagnosed cases of a new virus doubles every week. Let $C(t)$ be the number of diagnosed cases t weeks after an epidemic began. Here are the measurements during the first weeks:

t (weeks)	0	1	2	3	4	5
$C(t)$ (cases)	100	200	400	800	1600	3200

    (a) Mark the points corresponding to the data in the table on the graph of the function $C(t)$:



(b) Is the function $C = C(t)$ exponential? If yes, find a formula for $C(t)$ and check with the table of values.

5.3 Exponential Functions Numerically, Modeling

How do we recognize that a function given numerically — by a table of values — is an exponential function? This is an important question in applications. Typically when you study a real-life process you do not have a formula for a function describing the process ahead of time. You take measurements, tabulate the data, and then you try to find a mathematical model that fits your numerical data.

Fortunately, there is a simple test to check if a numerically given function is exponential or approximately exponential.

Example 5.3.1. Take the exponential function $y = f(x) = 3 \cdot 2^x$. Take its table of values for a few equally spaced values of x:

x	0	1	2	3	4	5
y	3	6	12	24	48	96

The values of x are equally spaced; that is, the difference between two consecutive values of x is constantly equal to $\Delta x = 1$:

$$1 - 0 = 1, \quad 2 - 1 = 1, \quad 3 - 2 = 1, \quad 4 - 3 = 1, \quad 5 - 4 = 1.$$

The differences of the corresponding y-values are, of course, not equal: $6 - 3 \neq 12 - 6$, $24 - 12 \neq 12 - 6$ and so on. (If the differences of consecutive values of y were equal, the table would represent a linear function.) Instead, the **ratios** of consecutive values of y are equal:

$$\frac{6}{3} = 2, \quad \frac{12}{6} = 2, \quad \frac{24}{12} = 2, \quad \frac{48}{24} = 2, \quad \frac{96}{48} = 2.$$

For every exponential function for equally spaced values of x, the ratios of the consecutive values of y are equal. We can see it easily from algebraic properties of exponential expressions. Take an exponential function $y = f(x) = A \cdot b^x$. Take a few equally spaced values for x. Denote them by x_1, x_2, x_3, x_4. They are equally spaced so for some Δx:

$$x_2 - x_1 = x_3 - x_2 = x_4 - x_3 = \Delta x.$$

Take the corresponding values of $f(x_1) = A \cdot b^{x_1}$, $f(x_2) = A \cdot b^{x_2}$, and so on, and look at the corresponding table:

x	x_1	x_2	x_3	x_4
y	$A \cdot b^{x_1}$	$A \cdot b^{x_2}$	$A \cdot b^{x_3}$	$A \cdot b^{x_4}$

Take the consecutive ratios of y-values and simplify them:

$$\frac{A \cdot b^{x_2}}{A \cdot b^{x_1}} = \frac{b^{x_2}}{b^{x_1}} = b^{x_2 - x_1} = b^{\Delta x},$$

$$\frac{A \cdot b^{x_3}}{A \cdot b^{x_2}} = \frac{b^{x_3}}{b^{x_2}} = b^{x_3 - x_2} = b^{\Delta x},$$

$$\frac{A \cdot b^{x_4}}{A \cdot b^{x_3}} = \frac{b^{x_4}}{b^{x_3}} = b^{x_4 - x_3} = b^{\Delta x}.$$

The consecutive ratios of y values are equal and they are all equal to $b^{\Delta x}$.

For every exponential function we have the equality of ratios. And vice-versa, if a table of values for $y = f(x)$ satisfies the constant ratios property, the data in the table can be modeled by an exponential function.

Note: The ratios are equal to $b^{\Delta x}$. Hence, if the values of x are spaced by 1 as in Example 5.3.1, that is $\Delta x = 1$, the ratios give b — the base of the exponential function. If x values are spaced by 2 or any other distance, so that $\Delta x \neq 1$, the ratios give b to some power and not the b itself. You have to be careful about that.

Example 5.3.2. Which of the functions given below are exponential? Give a formula for each function.

(a)

x	0	1	2	3	4	5
y	0.5	2	8	32	128	512

(b)

x	0	1	2	3	4	5
y	0.5	4.5	8.5	12.5	16.5	20.5

(c)

x	1	2	3	4	5
y	2	1	0.5	0.25	0.125

Solution. (a) Values of x are equally spaced: at each step x increases by 1. Thus, $\Delta x = 1$. We have to check if all ratios between consecutive values of y are the same:

$$\frac{2}{0.5} = 4, \quad \frac{8}{2} = 4, \quad \frac{32}{8} = 4, \quad \frac{128}{32} = 4, \quad \frac{512}{128} = 4.$$

Yes. All ratios are the same. Since $\Delta x = 1$, the ratios give us the growth factor $b = 4$. Of course, with each increase in x by 1, the current value of the function is multiplied by the constant factor of 4. The initial value, A is the value of the function at $x = 0$. The table shows that value to be 0.5. Hence, the table (a) corresponds to the exponential function $y = 0.5 \cdot 4^x$.

(b) Checking the ratios:

$$\frac{4.5}{0.5} = 9, \quad \frac{8.5}{4.5} = 1.888...$$

The first two ratios are not equal — not even close — so the function is not exponential. Observe that the values of the function start at 0.5 at $x = 0$ and then increase by the constant amount of 4 whenever x increases by 1. A function increasing by a constant amount — at a constant rate — is linear. We see easily that the data in the table (b) corresponds to the linear function $y = 0.5 + 4x$.

(c) Checking the ratios:

$$\frac{1}{2} = 0.5, \quad \frac{0.5}{1} = 0.5, \quad \frac{0.25}{0.5} = 0.5, \quad \frac{0.125}{0.25} = 0.5.$$

The ratios are equal and since x changes by 1 at each step, $\Delta x = 1$. Thus, the ratios give the value of the base $b = 0.5$. Hence, $y = A \cdot 0.5^x$. We still need the initial value A. We don't have the value of the function at 0 which gives A directly. Let's then set up an equation for A using, for example, the point $(1, 2)$ from the table. We have $A \cdot 0.5^1 = 2$. So $A \cdot 0.5 = 2$ which gives $A = 4$. The function represented by the table (c) is $y = 4 \cdot 0.5^x$.

Example 5.3.3. Check if $f(t)$ is an exponential function. If yes, find the formula for $f(t)$.

t	0	2	4	6	8	10
$f(t)$	0.3	0.9	2.7	8.1	24.3	72.9

Solution. Values of t are equally spaced by 2 units; that is, $\Delta t = 2$. We have to check if the consecutive ratios of function values are equal:

$$\frac{0.9}{0.3} = 3, \quad \frac{2.7}{0.9} = 3, \quad \frac{8.1}{2.7} = 3, \quad \frac{24.3}{8.1} = 3, \quad \frac{72.9}{24.3} = 3.$$

The ratios are all equal so $f(t) = A \cdot b^t$ is an exponential function. First notice that $A = f(0) = 0.3$. Hence, $f(t) = 0.3 \cdot b^t$ Let's try to find b. Is the base b equal to 3? No. At each step t changes by 2 units so the current value of $f(t)$ is multiplied by b twice. Better yet let's use a pair of values from the table to set up an equation for b. The table tells us that $f(2) = 0.9$. Hence:

$$\begin{aligned} f(2) &= 0.9 \\ 0.3 \cdot b^2 &= 0.9 \\ b^2 &= 0.9/0.3 \\ b^2 &= 3 \\ b &= \sqrt{3} \end{aligned}$$

Remember that the base b has to be positive so $b = \sqrt{3}$ rather than $b = \pm\sqrt{3}$. Our function is then $f(t) = 0.3 \cdot (\sqrt{3})^t$. We have $\sqrt{3} \approx 1.732$ so we can rewrite $f(t)$ as $f(t) = 0.3(1.732)^t$. We can also rewrite $f(t)$ as $f(t) = 0.3(3)^{\frac{t}{2}}$. The latter form shows that the values of the function triple every two units of t.

Modeling: The Population of Mozambique

Example 5.3.4. The population of Mozambique[2] between 2009 and 2014 is given below:

Year	2009	2010	2011	2012	2013	2014
Pop. in millions	22.89	23.53	24.19	24.86	25.56	26.28

Find a function that models Mozambique's population growth in the time period 2009-2014.

To simplify the task, denote by t the number of years since 2009 and by $P(t)$ the population at time t in millions. The table then becomes:

t	0	1	2	3	4	5
$P(t)$	22.89	23.53	24.19	24.86	25.56	26.28

Populations often increase exponentially so it is reasonable to check if $P(t)$ is exponential. We will round the ratios off to three decimal places. In real-life problems some rounding off is always necessary.

$$\frac{23.53}{22.89} = 1.028, \quad \frac{24.19}{23.53} = 1.028, \quad \frac{24.86}{24.19} = 1.028, \quad \frac{25.56}{24.86} = 1.028, \quad \frac{26.28}{25.56} = 1.028.$$

The ratios are equal so $P(t)$ is an exponential function. The growth factor is $b = 1.028$. Right from the table we obtain the initial value: $P(0) = A = 22.89$. The function that models the population growth in Mozambique is:

$$P(t) = 22.89(1.028)^t.$$

The growth factor is 1.028, the annual percent growth rate is 2.8%. The model is valid in the time period 2009-2014. Actually, if you examine the data given at worldometers.info website, the formula is valid in a longer time interval. The annual percent growth rate remains at approximately 2.8% between 2005 and 2020.

[2] https://www.worldometers.info/world-population/mozambique-population/, accessed: 3/10/2020

Practice Problems for Section 5.3

For each data table in Problems 1-5, check if the data can possibly correspond to an exponential function or not. If yes, give a formula for the function in the form $A \cdot b^t$. Identify the initial value and the growth factor.

1.

t	0	1	2	3	4	5
$f(t)$	5000	4000	3200	2560	2048	1638.4

2.

t	0	1	2	3	4	5
$g(t)$	32	48	72	108	162	243

3.

t	0	1	2	3	4	5
$k(t)$	64	32	16	8	0	2

4.

t	1	2	3	4	5
$h(t)$	1280	1600	2000	2500	3125

5.

t	0	1	2	3	4	5
$n(t)$	100	200	300	400	500	600

For each data table in Problems 6-7, check if the data can possibly correspond to an exponential function or not. If yes, give a formula for the function in the form $A \cdot b^t$. Give both the exact and the approximate value for the growth factor. Round off to at least three decimal places.

6.

t	0	2	4	6	8
$z(t)$	250	500	1000	2000	4000

7.

t	0	3	6	9	12	15
$m(t)$	64	32	16	8	4	2

In Problems 8-9, find a function $y = f(x)$ that corresponds to the data given in each table. Fill in the missing entries.

8.

x	0	1	2	3	4
y	500	900	1620	2916	?

9.

x	0	1	2	3	4
y	300	220	140	60	?

10. Barometric pressure[3] — the pressure of the air — depends on the altitude above sea level. Let H be altitude above sea level measured in kilometers. Let $P(H)$ be barometric pressure measured in mmHg — millimeters of mercury. Here are readings of barometric pressure at different altitudes:

[3] https://en.wikipedia.org/wiki/Barometric_formula, accessed: 7/10/20

H (km)	0	1	2	3	4
$P(H)$ (mmHg)	760	674.12	597.94	530.37	470.44

(a) Can you see from the data that the function $P(H)$ is exponential?

(b) What is the barometric pressure, P_0, at sea level?

(c) Find a formula for $P(H)$ in the form $P(H) = P_0 \cdot b^H$. When calculating b, round off to three decimal places.

(d) The summit of Mount Everest is at 8848 meters above sea level. What is the barometric pressure at the top of Mount Everest?

5.4 Doubling Time and Half-Life

In this section we introduce two important concepts associated with exponential processes: the doubling time of a process of exponential growth and the half-life of a process of exponential decay.

The Doubling Time

In Example 5.1.2 we looked at a population of E.coli bacteria that grows exponentially according to the formula:

$$f(t) = 2000(1.0353)^t,$$

where t is measured in minutes, $f(t)$ in the number of bacteria. We noticed that the population doubles every 20 minutes:

$$f(20) \approx 4000, \quad f(40) \approx 8000, \quad f(60) \approx 16000.$$

For every exponentially increasing function the time needed for the current value to double is constant and it is called the *doubling time*.

The Doubling Time

Let $f(t) = A \cdot b^t$, $b > 1$, be an increasing exponential function. The time needed for the value $f(t)$ to double is called the **doubling time** of the function f.

Example 5.4.1. Show algebraically that every increasing exponential quantity $f(t) = A \cdot b^t$, $b > 1$, has a fixed doubling time; that is, it always takes the same amount of time for the quantity $f(t)$ to double.

Solution. Let t_d be the time after which the initial amount $f(0) = A$ doubles. In other words, after time t_d, the value $f(t_d) = A \cdot b^{t_d}$ is twice the initial value A:

$$A \cdot b^{t_d} = A \cdot 2.$$

We divide both sides by A and see that t_d is such a time that:

$$b^{t_d} = 2.$$

But then the quantity doubles again after another t_d units of time:

$$f(t_d + t_d) = A \cdot b^{t_d + t_d} = A \cdot b^{t_d} \cdot b^{t_d} = A \cdot 2 \cdot 2.$$

In fact, no matter what time t we start from, after t_d units of time the quantity $f(t)$ will double:

$$f(t + t_d) = A \cdot b^{t + t_d} = A \cdot b^t \cdot b^{t_d} = f(t) \cdot 2$$

as $A \cdot b^t = f(t)$ and $b^{t_d} = 2$.

It is important to realize that only exponentially increasing quantities have a doubling time. Processes modeled by different functions do not.

Example 5.4.2. Consider a function $g(t) = t^2 + 4$. Show that $g(t)$ does not have a fixed doubling time.

Solution. The initial value $g(0) = 4$. How much time does it take for the value 4 to double? We need to find a positive t such that $g(t) = t^2 + 4 = 2 \cdot 4 = 8$. That is:

$$t^2 + 4 = 8.$$

Solving the equation we get $t = 2$. It takes 2 units of time for the initial amount to double. How long does it take for the quantity $g(t)$ to double again from 8 to 16? It is for for $t = \sqrt{12} \approx 3.46$ that:

$$t^2 + 4 = 16$$

Hence, it took 2 units of time for $g(t)$ to double from 4 to 8 but it takes $3.46 - 2 = 1.46$ units of time after that to double again from 8 to 16. The time needed for $g(t)$ to double is not constant. $g(t)$ does not have a fixed doubling time. Of course not: $g(t)$ is not an exponential function.

Example 5.4.3. You deposit \$3000 into a savings account that pays 4.43% interest annually. Let $B = B(t)$ be your balance after t years. As we saw in Section 5.1, the formula for $B(t)$ is:

$$B(t) = 3000(1.0443)^t.$$

How long will it take for your money to double?

Solution. The balance function $B(t)$ is an increasing exponential function. We want to find its doubling time; that is, we want to find t after which the initial amount doubles:

$$3000(1.0443)^t = 2 \cdot 3000.$$

We divide both sides by 3000. The equation becomes:

$$1.0443^t = 2.$$

We cannot solve this equation algebraically as the unknown, t, is in the exponent. Solving such equations requires logarithms which we will study in the next chapter.

We can find an approximate solution by graphing the function $y = 1.0443^x$ using a graphing calculator and then finding the point on the graph where $y = 2$. Another way of finding the doubling time approximately is to evaluate the function $B(t) = 3000(1.0443)^t$ for a few suitable values of t:

t	10	15	16	17	32
$B(t)$	4627.79	5747.78	6002.41	6268.32	12009.64

It takes approximately 16 years for your initial deposit \$3000 to double to \$6000 — the doubling time is approximately 16 years. Note that after the next 16 years when $t = 32$ the balance \$6000 doubles again and so on. We can predict that after 48 years the balance will be about 24000.

If the base of an increasing exponential function is written in terms of powers of 2, we can find the doubling time algebraically without logarithms.

Example 5.4.4. The value $V(t)$, in dollars, of an investment t years after an initial amount of \$2000 was invested is given by

$$V(t) = 2000 \cdot 2^{t/10}.$$

(a) How much will the investment be worth for $t = 10$? $t = 20$? $t = 30$? What pattern do you observe unfolding?

(b) What is the doubling time of the investment?

(c) Write the expression for $V(t)$ in the form $A \cdot b^t$. Identify the growth factor and the annual percent growth rate.

Solution. (a) We calculate:

$$V(10) = 2000 \cdot 2^{10/10} = 4000,$$

$$V(20) = 2000 \cdot 2^{20/10} = 2000 \cdot 2^2 = 8000,$$
$$V(30) = 2000 \cdot 2^{30/10} = 2000 \cdot 2^3 = 16000.$$

The investment doubles every 10 years.

(b) As we saw in (a), the doubling time of $V(t)$ is 10 years, $t_d = 10$.

(c) To rewrite $V(t)$ in the form $V(t) = A \cdot b^t$, observe that

$$V(t) = 2000 \cdot 2^{t/10} = 2000 \cdot (2^{1/10})^t.$$

We use our calculator and calculate $2^{1/10} = 1.072$. We rounded to the third place after the decimal point. Now we can write the function $V(t)$ in the standard form:

$$V(t) = 2000 \cdot 1.072^t.$$

The growth factor b is 1.072. The annual percent growth rate r is 7.2% as $r = b - 1 = 1.072 - 1 = 0.072$. (In other words, $b = 1 + 0.072$.)

As the example above illustrates, if an exponential function is written in terms of powers of 2, finding the doubling time is very easy.

A General Rule:

For an exponential function $f(t)$ written in the form:

$$f(t) = A \cdot 2^{\frac{t}{T}}$$

the doubling time is T. The growth factor is

$$b = 2^{\frac{1}{T}}.$$

The Half-Life

A **decreasing** exponential function: $f(t) = A \cdot b^t$, $b < 1$, doesn't have a doubling time — it never doubles as it is decreasing. A decreasing exponential function has the so-called half-life. The half-life of $f(t)$ is the time needed for the value of the function to be halved. For an exponentially decaying quantity the time needed for the quantity to be reduced by half is constant and it is called the *half-life*. Let's denote the half-life by t_h. After the time t_h, the initial value will be reduced by half, after the next t_h units of time the value of the function will be halved again and so on. Half-life is exceptionally important in pharmacology. When you read leaflets that come with medications, they always give you the half-life of a medication — the time needed for the half of the medication in your body to be gone following a dose.

The Half-Life

Let $f(t) = A \cdot b^t$, $b < 1$, be an decreasing exponential function. The time needed for the value $f(t)$ to be reduced by half is called the **half-life** of the function f.

Again, before we study logarithms, we cannot find algebraically the half-life for an exponential function unless the base of the function is expressed in terms of powers of $\frac{1}{2}$.

Example 5.4.5. Let $Q(t) = 100\left(\frac{1}{2}\right)^{t/5}$ be the amount of a medication in a patient's bloodstream, in mg, t hours after a single dose of 100 mg.

(a) How much of the medication is left in the patient's bloodstream after 5 hours? 10 hours? 15 hours? What pattern do you observe unfolding? What is the half-life of the medication?

(b) Rewrite $Q(t)$ in the form $Q(t) = A \cdot b^t$ for constants A and b. Give the growth factor and the hourly percent growth rate.

Solution. (a) We calculate:

$$Q(5) = 100\left(\frac{1}{2}\right)^{5/5} = 100\left(\frac{1}{2}\right) = 50,$$

$$Q(10) = 100\left(\frac{1}{2}\right)^{10/5} = 100\left(\frac{1}{2}\right)^2 = 25,$$

$$Q(15) = 100\left(\frac{1}{2}\right)^{15/5} = 100\left(\frac{1}{2}\right)^3 = 12.5.$$

Every 5 hours the amount of the medication left in the bloodstream is cut by half. Hence, the half-life of the medication is 5 hours, $t_h = 5$.

(b) We rewrite the formula for the function as: $Q(t) = 100\left(\left(\frac{1}{2}\right)^{1/5}\right)^t$. We calculate: $\left(\frac{1}{2}\right)^{1/5} \approx 0.871$. We rounded to the third decimal place. Hence:

$$Q(t) = 100 \cdot 0.871^t.$$

The growth factor is 0.871. (We remember that the base b is called "the growth factor" even for decreasing exponential functions $f(t) = A \cdot b^t$.) The hourly percent growth rate is -12.9% as $0.871 - 1 = -0.129$. We can also say that the hourly percent decay rate is 12.9%.

As the example above illustrates, if an exponential function is written in terms of powers of $\frac{1}{2}$, finding the half-life is very easy.

A General Rule:

For an exponential function $f(t)$ written in the form:

$$f(t) = A \cdot \left(\frac{1}{2}\right)^{\frac{t}{T}}$$

the half-life is T. The growth factor is

$$b = \left(\frac{1}{2}\right)^{\frac{1}{T}}.$$

Example 5.4.6. A patient takes a 40 mg tablet of a common anti-heartburn medication Famotidine. The amount of the medication in the bloodstream, $F(t)$, in mg, t hours after the dose, decays exponentially with the half-life 3.8 hours.

(a) Find a formula for $F(t)$.

(b) Find the growth factor and the percent decay rate.

Solution. (a) We have the initial amount $A = 40$ and we have the half-life. We can write a formula for $F(t)$ in terms of powers of $\frac{1}{2}$:

$$F(t) = 40 \cdot \left(\frac{1}{2}\right)^{\frac{t}{3.8}}.$$

(b) To rewrite $F(t)$ in the form $F(t) = 40 \cdot b^t$, notice that:

$$F(t) = 40 \cdot \left(\left(\frac{1}{2}\right)^{\frac{1}{3.8}}\right)^t.$$

Hence, the growth factor $b = \left(\frac{1}{2}\right)^{\frac{1}{3.8}} = 0.833$ and $F(t) = 40(0.833)^t$. (We rounded off b to three decimal places.) The percent growth rate is $0.833 - 1 = -0.167 = -16.7\%$.

Additional Observations and Examples

Example 5.4.7. Let $P(t)$ be the population of a village, in the number of people, t years after the village was founded. The population that was 250 people initially triples every 15 years.

(a) Find a formula for $P(t)$.

(b) Write $P(t)$ in the form $P(t) = A \cdot b^t$.

Solution. (a) If we take $P(t) = 250 \cdot 3^t$, the population is multiplied by the factor of 3 every year. To have the population multiplied by 3 every 15 years, we should take:

$$P(t) = 250 \cdot 3^{\frac{t}{15}}.$$

Of course, with $P(t)$ as above we have: $P(15) = 250 \cdot 3^{\frac{15}{15}} = 250 \cdot 3 = 750$,
$P(30) = 250 \cdot 3^{\frac{30}{15}} = 250 \cdot 3^2 = 750 \cdot 3 = 2250$, and so on. The population triples every 15 years.

(b) We rewrite:

$$P(t) = 250 \cdot 3^{\frac{t}{15}} = 250 \cdot (3^{\frac{1}{15}})^t$$

We calculate $3^{\frac{1}{15}} \approx 1.076$. Hence:

$$P(t) = 250 \cdot 1.076^t.$$

We make the following general observation:

A General Rule:

For an exponential function of the form

$$f(t) = A \cdot c^{\frac{t}{T}},$$

A is the initial quantity and c is the factor by which the quantity increases or decreases every T units of time.

Example 5.4.8. Find a formula $V(t) = A \cdot b^t$ for the value of an investment initially worth 1500 that grows 12% every 6 years.

Solution. A quantity grows by 12% when it is multiplied by the factor of 1.12. Hence, $V(t)$ is multiplied by 1.12 every 6 years. According to the rule:

$$V(t) = 1500 \cdot 1.12^{\frac{t}{6}}.$$

To get $V(t)$ in the form $V(t) = A \cdot b^t$, note that:

$$V(t) = 1500 \cdot 1.12^{\frac{t}{6}} = 1500 \cdot (1.12^{\frac{1}{6}})^t = 1500(1.0191)^t.$$

Practice Problems for Section 5.4

In each of the Problems 1-4 an exponential function, $y = f(t)$, is given. For each of the functions, find the initial amount (in units of y) and the doubling time (in units of t).

1. $y = 47 \cdot 2^{\frac{t}{5}}$
2. $y = 800 \cdot 2^{\frac{t}{9}}$
3. $y = 89 \cdot 4^{\frac{t}{6}}$
4. $y = 120 \cdot 8^{\frac{t}{12}}$

In each of the Problems 5-8 an exponential function, $y = f(t)$, is given. For each of the functions, find the initial amount (in units of y) and the half-life (in units of t).

5. $y = 75 \cdot \left(\frac{1}{2}\right)^{\frac{t}{7}}$
6. $y = 920 \cdot \left(\frac{1}{2}\right)^{\frac{t}{10}}$
7. $y = 875 \cdot \left(\frac{1}{16}\right)^{\frac{t}{12}}$
8. $y = 63 \cdot \left(\frac{1}{4}\right)^{\frac{t}{10}}$

In Problems 9-12: Write a formula for an exponential function in the form $f(t) = A \cdot c^{\frac{t}{T}}$ that describes a given scenario. Then convert the function to the form $f(t) = A \cdot b^t$.

9. The population of a town begins with 1500 people at $t = 0$ and doubles every 8 years.

10. The population of a town begins with 750 people at $t = 0$ and triples every 9 years.

11. The initial amount of 50 mg of a radioactive element is halved every 11 days.

12. The initial amount of 60 mg of a radioactive element is cut by one-third every 5 months.

In Problems 13-14 estimate the half-life or the doubling time whichever applies.

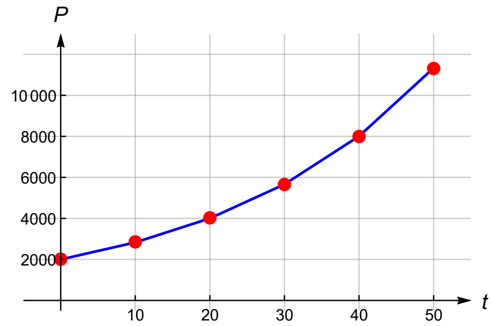
13.

t	0	1	2	3	4	5	6	7
$f(t)$	100.52	126.65	159.57	201.04	253.29	319.13	402.08	506.59

14.

t	0	1	2	3	4	5	6	7
$g(t)$	20.15	14.25	10.08	7.12	5.04	3.56	2.52	1.78

15. The graph of an exponential function below shows the population of bacteria, $P(t)$, in a laboratory experiment t minutes after the experiment began.

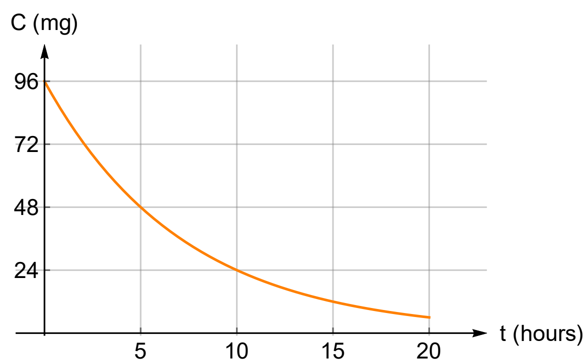


(a) What is the initial amount, P_0, of bacteria?

(b) Estimate the doubling time of the population.

(c) Write a formula for $P(t)$ in the form $P(t) = P_0 \cdot 2^{\frac{t}{T}}$. Then rewrite the formula in the form $P(t) = P_0 \cdot b^t$. Round off b to four decimal places. What is the percent growth rate?

16. The amount of caffeine remaining in the body, $C = C(t)$, in milligrams, t hours after drinking a cup of coffee, is an exponential function and its graph is given below:



(a) Estimate the amount of caffeine, C_0, absorbed by the body from a cup of coffee.

(b) Estimate the half-life of caffeine.

(c) Write a formula for $C(t)$ in the form $C(t) = C_0 \cdot \left(\frac{1}{2}\right)^{\frac{t}{T}}$. Rewrite the formula in the form $C(t) = C_0 \cdot b^t$. Round b off to three decimal places. What is the percent growth rate?

5.5 The Natural Base "e"

A commonly used base in the context of exponential functions is the *Euler constant*. The constant is denoted by the letter e and its approximate value is:

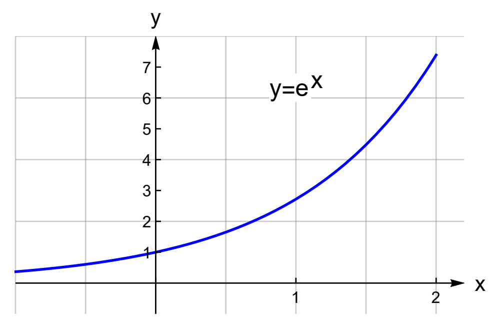
$$e \approx 2.718281827...$$

e is an irrational number which, much like π, keeps popping up in various areas of mathematics. It is sufficient for you to remember that $e \approx 2.718$ or even

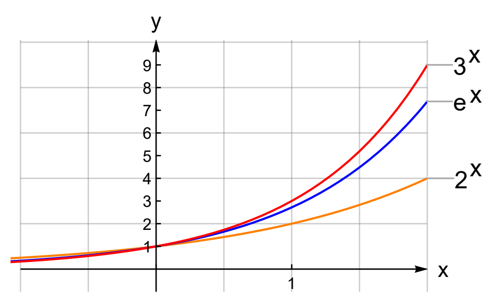
$$e \approx 2.71.$$

Your calculator will always give you a better approximation of e. A precise definition of e is complicated and we will not study it here.

e is called the **natural base**. The exponential function $y = e^x$ is called the natural exponential function. Since e is a number like any other and since $e > 1$, the graph of an exponential function has the same shape as the graphs of other increasing exponential functions:



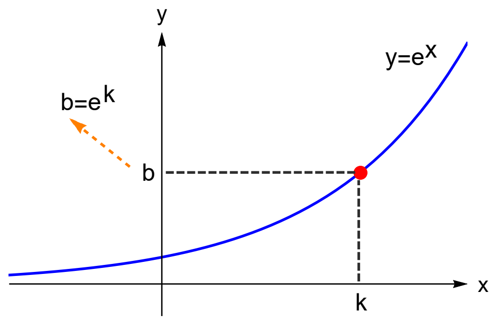
Since $2 < e < 3$, the graph of $y = e^x$ is between the graphs of $y = 2^x$ and $y = 3^x$:



In most applications, especially in life sciences, people tend to rewrite **all** exponential functions in terms of the natural base e. How can this be done? Suppose we have an exponential function $f(x) = A \cdot b^x$. Any base b, $b > 0$, can be rewritten in terms of e. Namely, for any positive b, there is a constant k such that:

$$b = e^k.$$

It is clear geometrically if we look at the graph of the natural exponential:



If $b = e^k$, we can rewrite any exponential function as:

$$f(x) = A \cdot b^x = A \cdot (e^k)^x = A \cdot e^{kx}.$$

We can also see from the graph that for $b > 1$, the constant k is positive. For $b < 1$, the constant k is negative. In general:

Exponential Function in Terms of Base e

Any exponential function $Q(t) = A \cdot b^t$ can be written in the form

$$Q(t) = A \cdot e^{kt}$$

where k is such a constant that $e^k = b$. We have:

- If $b > 1$, then $k > 0$.
- If $b < 1$, then $k < 0$.
- The constant k is called the *continuous growth rate.*
- We can convert from $Q(t) = A \cdot e^{kt}$ form to the form $Q(t) = A \cdot b^t$ by setting the growth factor $b = e^k$.

Remark: We adopt the standard terminology and call the constant k in the formula $Q(t) = A \cdot e^{kt}$ the "continuous growth rate" or sometimes, especially if k is expressed in terms of percents, the "continuous percent growth rate". However, at this point, before studying Calculus, we cannot explain the meaning of the term "continuous growth rate" in any practical or precise way. For now, we will simply accept the name "continuous growth rate" for the constant k and we will know that whenever we are given a continuous growth rate, we are given the constant k that appears in the formula $y = A \cdot e^{kt}$.

It is important to keep in mind the following facts.

- The value of the constant k depends on the units of t — years, days, hours etc. Hence, we will sometimes say the "continuous annual growth rate" k or the "continuous daily growth rate" k etc., to emphasize the units of t unless the units of t are clear from the context.

- We can and we will often compare the values of the continuous percent growth rate k and the percent growth rate r that appears in the formula:

$$Q(t) = A \cdot b^t = A(1+r)^t.$$

The constants r and k are different although their values are often close.

Note: If we have an exponential function in base-e form: $Q(t) = A \cdot e^{kt}$, we can convert it easily to standard form $Q(t) = A \cdot b^t$ by taking $b = e^k$. Going the other way around, that is, finding the exact value of k for a given b requires solving the following equation for k:

$$e^k = b.$$

Since the unknown k is in the exponent we cannot solve the equation algebraically until Chapter 6 as solving it requires logarithms.

Example 5.5.1. The value of investment, $V(t)$, is given by:

$$V(t) = 2500e^{0.035t}$$

where t is in years. Give the initial value and the continuous growth rate. Rewrite the function in the form:

$$V(t) = A \cdot b^t.$$

Give the growth factor and the percent growth rate. Compare the percent growth rate and the continuous percent growth rate.

Solution. From the base-e form, we see that the **continuous** growth rate $k = 0.035 = 3.5\%$. The initial value of the investment is \$2500.

Now notice that:

$$V(t) = 2500e^{0.035t} = 2500\left(e^{0.035}\right)^t = 2500 \cdot 1.0356^t$$

as $e^{0.035} \approx 1.0356$. Hence, the growth factor is 1.0356, and the annual percent growth rate is $0.0356 = 3.56\%$. We see that the continuous growth rate and the annual percent growth rate are close but not quite the same.

Example 5.5.2. A patient treated for thyroid cancer is given an injection of 10 μg of Iodine-131. Let $I(t)$ be the amount of Iodine-131 left in the patient's body t days after the injection. Given that the **continuous** daily decay rate of Iodine-131 is 8.66% — in other words, the continuous daily growth rate is -8.66% — find:

(a) A formula for $I(t)$.

(b) Convert the formula to the form $I(t) = 10 \cdot b^t$. Compare the daily percent growth rate and the continuous growth rate.

Solution. (a) We are given the **continuous** growth rate $k = -8.66\% = -0.0866$. Hence, it will be easier to find a formula for $I(t)$ in the form $I(t) = Ae^{kt}$ as we are given k. The initial amount $A = 10$ so:

$$I(t) = 10e^{-0.0866t.}$$

(b) To convert to the form $I(t) = 10 \cdot b^t$, take $b = e^k = e^{-0.0866} = 0.917$. We have:

$$I(t) = 10(0.917)^t.$$

The daily percent growth rate is $r = 0.917 - 1 = -0.083 = -8.3\%$.

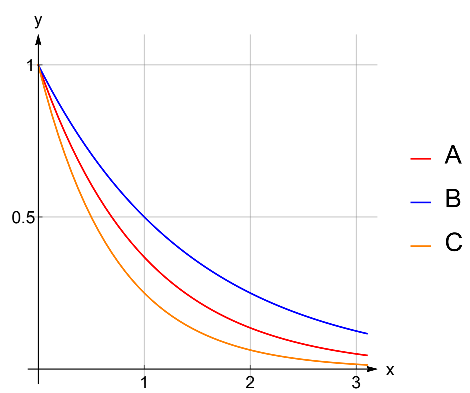
The two rates, -8.66% and -8.3%, differ significantly.

Practice Problems for Section 5.5

In Problems 1-4 use your calculator to evaluate given expressions. Round off your answers to three decimal places.

1. e^2
2. $e^{1.5}$
3. e^{-1}
4. $\dfrac{e^3+1}{e^2}$

5. Match the graphs (A), (B), (C) given below with the following functions:
   (i) $y = 4^{-x}$ (ii) $y = 2^{-x}$ (iii) $y = e^{-x}$.



In Problems 6-9 determine if a given exponential function is increasing or decreasing. For each function, identify its continuous growth rate.

6. $y = 100e^{0.12x}$
7. $y = 1200e^{-0.029x}$
8. $y = 20e^{0.065x}$
9. $y = 700e^{-0.38x}$

In Problems 10-13 rewrite a given exponential function in the form $y = A \cdot b^t$. Round off the base b to four decimal places.

10. $y = 120e^{0.32t}$

11. $y = 1500e^{-0.098t}$

12. $y = 200e^{-0.16t}$

13. $y = 1700e^{0.09t}$

14. A common antidepressant Paxil has a continuous hourly growth rate of -3.3%. A patient takes an initial dose of 30 mg. Let $P(t)$ be the amount of Paxil left in a patient's body from the initial dose t hours later.

    (a) Write a formula for $P(t)$ in terms of the natural base e.

    (b) How much Paxil is left in the patient's body 18 hours after the dose?

15. A radioactive isotope of Iodine, Iodine-123, is often used in medical imagining as a contrast. An initial amount of 12 μg of Iodine-123 is administered to a patient. Let $I(t)$ be the amount of Iodine-123, in μg, remaining in the patient's body after t hours. Given that the continuous hourly decay rate of Iodine-123 is 5.33%, write a formula for $I(t)$. How much Iodine-123 is left after 24 hours?

Chapter 6

Logarithms and Their Applications

6.1 What Are Logarithms?

When studying processes of exponential growth and decay in Chapter 5 we encountered equations of the type:

$$3000(1.0433)^t = 6000$$

where the unknown is in the exponent. To solve such equations algebraically and for many other reasons we need *logarithms*.

> **Definition Of Logarithm Base b**
>
> Let b such that $b > 0$, $b \neq 1$ be given. Then for every $x > 0$, the logarithm base b of x, denoted $\log_b x$, is such a number that:
>
> $$b^{\log_b x} = x.$$

$\log_b x$ is such a number, such an **exponent**, that the base b raised to this exponent is x. The following picture makes it easy to remember:

$$\log_b x = p \quad \Leftrightarrow \quad b^p = x$$

In other words, $\log_b x$ is equal to such a number p that b to that number is x.

The expressions

$$\log_b x = p \quad \text{and} \quad b^p = x$$

are equivalent. They are a logarithmic and an exponential version of the same statement.

b is called the base of $\log_b x$. $\log_b x$ is called the "logarithm base b" or the "logarithm to the base b" or the "logarithm with the base b".

Example 6.1.1. (a) $\log_2 8 =?$

Answer: We have the logarithm base 2 in this example. $\log_2 8$ is such a number — such an exponent — that 2 to this exponent is equal 8. This exponent is, of course, 3:

$$\log_2 8 = 3 \quad \text{as} \quad 2^3 = 8.$$

Let's look at our circle:

$$\log_2 8 = 3 \quad \text{as} \quad 2^3 = 8.$$

(b) $\log_3 \frac{1}{9} = ?$

Answer: The logarithm base 3 of $\frac{1}{9}$, $\log_3 \frac{1}{9}$, is such a number — such an exponent — that 3 to this exponent is equal $\frac{1}{9}$. This exponent is, of course, -2:

$$\log_3 \frac{1}{9} = -2 \quad \text{as} \quad 3^{-2} = \frac{1}{3^2} = \frac{1}{9}.$$

(c) $\log_{10} 0.001 = ?$

Answer: The logarithm with the base 10 of $0.001 = \frac{1}{1000}$ is such an exponent that 10 to that exponent is equal to $\frac{1}{1000}$. We have

$$10^{-3} = \frac{1}{10^3} = \frac{1}{1000} = 0.001.$$

Hence:

$$\log_{10} 0.001 = -3.$$

Graphically:

$$\log_{10} 0.001 = -3.$$

(d) $\log_e(e^2) = ?$

Answer: The logarithm to the base e of e^2, $\log_e(e^2)$, is such an exponent that e to that exponent is e^2. Thus:

$$\log_e(e^2) = 2.$$

Note: If the argument under the logarithm is a more complicated expression rather than just a number or a letter, we use parentheses around it. Often we use parentheses even if the argument is just a number or a letter.

Example 6.1.2. (a) $\log_{10}(-100) = ?$

Answer: We are looking for a number such that 10 to that number is equal to -100:

$$10^{?} = -100$$

Would it be -2? Definitely not as $10^{-2} = \frac{1}{100}$. There is no number such that 10 to that number is -100 as 10^p is always positive by properties of exponential functions.

(b) $\log_e 1 = ?$

Answer: We are asking: $e^{?} = 1$. We have $e^0 = 1$. Hence, $\log_e 1 = 0$.

The last example illustrates the following properties of logarithms which hold for any base b:

- If w is negative or 0, $\log_b w$ is undefined as $b^p > 0$ for any power p.
- $\log_b 1 = 0$ as $b^0 = 1$ for any b.

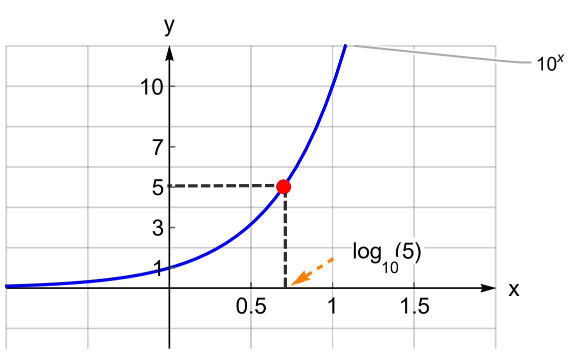
Example 6.1.3. Find:

$$\log_{10}(5) = ?$$

Answer: We are looking for an exponent such that 10 to this exponent is equal to 5:

$$10^{?} = 5$$

Notice $10^0 = 1$ so 0 is too small. $10^1 = 10$ so 1 is too large. The exponent, the number, that we are looking for is somewhere between 0 and 1. There is such a number. Indeed, look at the graph of $y = 10^x$:



Clearly there is a number between 0.5 and 1 such that 10 to that number is 5. That number is by definition $\log_{10}(5)$. What's the simplest way to find it? Use your calculator! Most likely you have two buttons on your calculator related to logarithms: "log" and "ln". "log" is the logarithm to the base 10. We use our calculator and obtain:

$$\log_{10}(5) \approx 0.69897$$

There are essentially two logarithms that are important in applications. The first one is the logarithm to the base 10. The logarithm to the base 10 is denoted by $\log(x)$ and called the *common logarithm*:

$\log(x) = \log_{10}(x)$ - the **common logarithm**.

The second logarithm important in applications is the logarithm to the base e denoted by $\ln(x)$ and called the *natural logarithm*:

$\ln(x) = \log_e(x)$ - the **natural logarithm**.

The logarithm to the base 2 has some importance in the theory of music and other applications but we will focus on the common and the natural logarithms.

To evaluate and manipulate these two logarithms correctly, you have to remember what their bases are:

$$\log x = \log_{10} x = p \quad \Leftrightarrow \quad 10^p = x$$

$$\ln x = \log_e x = p \quad \Leftrightarrow \quad e^p = x$$

Example 6.1.4. Evaluate the following expressions without a calculator.

(a) $\ln e$

Answer: The natural logarithm "ln" is the logarithm to the base e. Since $e^1 = e$, we have $\ln e = 1$.

(b) $\ln\left(\frac{1}{e^3}\right)$

Answer: Let's simplify the expression under the logarithm:

$$\left(\frac{1}{e^3}\right) = e^{-3}$$

Hence:

$$\ln\left(\frac{1}{e^3}\right) = \ln(e^{-3}) = -3.$$

(c) $2\ln\left(\frac{\sqrt{e}}{e}\right)$

Answer: Again, let's simplify the expression under the logarithm and then evaluate the logarithm and the expression:

$$2\ln\left(\frac{\sqrt{e}}{e}\right) = 2\ln\left(\frac{e^{1/2}}{e}\right) = 2\ln\left(e^{1/2-1}\right) = 2\ln(e^{-1/2}) = 2\left(-\frac{1}{2}\right) = -1.$$

(d) $\ln 0$

Answer: Undefined. The natural logarithm as any other logarithm is defined for positive inputs only. Of course, there is no number p such that $e^p = 0$. e^p is always positive.

(e) $\ln 1$

Answer: $\ln 1 = 0$ as $e^0 = 1$.

(f) $\ln(e^{-0.078})$

Answer: We have to raise e to the power -0.078 to get $e^{-0.078}$. Hence, $\ln(e^{-0.078}) = -0.078$.

(g) $\log(\log 10)$

Answer: $\log 10 = 1$. Hence, $\log(\log 10) = \log(1) = 0$.

(h) $\log(\sqrt{10}) - \log(10^2)$

Answer: We evaluate each term and then take a difference: $\log(\sqrt{10}) = \log(10^{\frac{1}{2}}) = \dfrac{1}{2}$; $\log(10^2) = 2$. Hence:

$$\log(\sqrt{10}) - \log(10^2) = \frac{1}{2} - 2 = -\frac{3}{2}$$

Example 6.1.5. Use your calculator to find the following logarithms.

(a) $\ln 2$

Answer: We use the "ln" button of your calculator and obtain $\ln 2 \approx 0.693$. We rounded off to three decimal places. As many logarithms, $\ln 2$ is an irrational number so we can only have its decimal approximation. Let's check: $e^{0.693} \approx 1.99970$ — it is almost 2.

(b) $\log 2$

Answer: We use the "log" button of your calculator and obtain $\log 2 \approx 0.301$. We rounded off to three decimal places. Let's check: $10^{0.301} \approx 1.99986$ — practically 2.

(c) $\log(e^{-0.078})$

Answer: We use a calculator and obtain: $\log(e^{-0.078}) \approx -0.0339$. Check: $10^{-0.0339} \approx 0.9249$, $e^{-0.078} \approx 0.9249$. Yes, we calculated the logarithm correctly.

Graphs of Functions $y = \log(x)$ and $y = \ln(x)$

The common logarithm $\log(x)$ is defined for every positive input x and so is the natural logarithm $\ln(x)$. We can consider then the two logarithmic functions $y = \log(x)$ and $y = \ln(x)$ with domains $x > 0$. The graphs of the common logarithm function and the natural logarithm function have a similar shape:

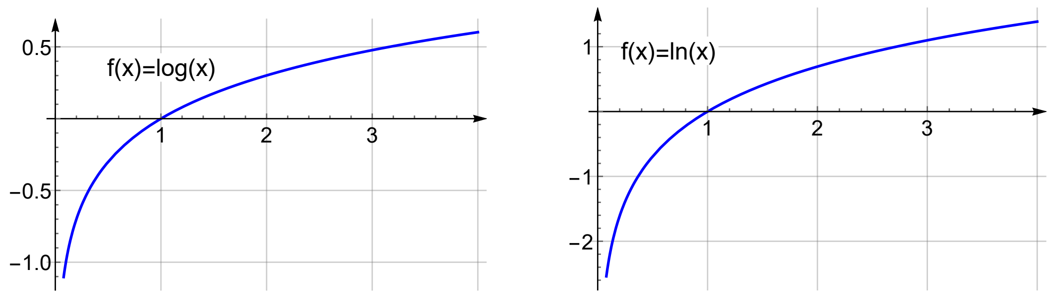


Figure 6.1

We see that both functions $y = \log x$ and $y = \ln x$ are defined for all $x > 0$ and not defined for $x \leq 0$. Both functions are 0 at $x = 1$: $\log 1 = 0$, $\ln 1 = 0$. Both functions are positive for $x > 1$ and negative for $0 < x < 1$. We also notice that for positive inputs x which are closer and closer to 0, outputs are getting "more and more negative" and both graphs "snuggle" to the y-axis. We say that the y-axis is a vertical asymptote for $y = \log x$ and $y = \ln x$. Both functions are increasing and for both functions to different inputs there correspond different outputs.

Practice Problems for Section 6.1

In Problems 1-18 evaluate a given expression without a calculator. If the expression is undefined, say so.

1. $\log_2\left(\frac{1}{64}\right)$
2. $\log_2\left(-\frac{1}{16}\right)$
3. $\frac{1}{2}\log_2(0)$
4. $\log_3(27)$
5. $\log_3\left(\frac{1}{9}\right)$
6. $\log_3\left(\frac{\sqrt{3}}{3}\right)$
7. $2\ln(e^{12})$
8. $\ln(\ln e)$
9. $\frac{\ln(1)}{\ln(e^2)}$
10. $\frac{\ln(e^2)}{\ln(1)}$

11. $\ln(\sqrt{e}) + \ln(e^3)$

12. $\log\left(\frac{1}{10^4}\right)$

13. $\log(\sqrt[3]{10}) - 3\log\left(\frac{10}{\sqrt{10}}\right)$

14. $\log(10 - 10^2)$

15. $\frac{3\ln(e^{-2})}{\ln(\sqrt{e})}$

16. $\frac{\ln(-e)}{\ln(e^2)}$

17. $\ln(e^2 e^3)$

18. $\log(100^2)$

In Problems 19-22 use the graphs of $y = \ln(x)$ and $y = \log(x)$ in Figure 6.1 to give a rough estimate of:

19. $\ln(0.2)$

20. $\ln(1.2)$

21. $\log(0.2)$

22. $\log(2)$

In Problems 23-26 use your calculator to evaluate the following expressions. Round off to four decimal places. For undefined expressions, state "undefined".

23. $\log(7)$

24. $\frac{\ln(2)}{0.327}$

25. $\frac{\log(5)}{2\log(7)}$

26. $\frac{\ln(0.5)}{-0.0278}$

6.2 Properties of Logarithms, Solving Equations Using Logarithms

Algebraic properties of logarithms is what makes them useful in applications. Those properties are similar for all logarithms with all bases b. For the sake of clarity, though, we will list and practice separately the properties of the common logarithm and the properties of the natural logarithm.

Algebraic Properties of Logarithms

We begin with the common logarithm.

Properties of the Common Logarithm

For A and B positive numbers:

1. $\log(10^x) = x \quad$ for any x
2. $10^{\log x} = x \quad$ for x positive
3. $\log(A \cdot B) = \log A + \log B$
4. $\log(\frac{A}{B}) = \log A - \log B$
5. $\log(B^p) = p \log B \quad$ for any p

Properties 1 and 2 follow easily from the definition of the logarithm to the base 10. We practiced them in the previous section.

Why does Property 3 hold? Why is the logarithm of a product equal to the sum of logarithms? It is very easy to explain. From the definition of $\log x = \log_{10} x$ we know that for any number or any expression — to keep it light let's denote the expression by $\heartsuit$ — we have:

$$\log(A \cdot B) = \heartsuit \quad \Leftrightarrow \quad 10^{\heartsuit} = A \cdot B.$$

Notice that from properties of exponential expressions we have:

$$10^{\log A + \log B} = 10^{\log A} \cdot 10^{\log B}$$

Indeed, the sum of exponents corresponds to the product of the exponential expressions. From Property 2 of the common logarithm:

$$10^{\log A} = A, \quad 10^{\log B} = B.$$

We have then:

$$10^{\log A + \log B} = 10^{\log A} \cdot 10^{\log B} = A \cdot B.$$

Hence:

$$\log(A \cdot B) = \log A + \log B \quad \text{as} \quad 10^{\log A + \log B} = A \cdot B.$$

Similarly, Property 4 and 5 of the logarithm follow easily from the algebraic properties of exponential expressions.

All logarithms have similar properties as those listed above for the common logarithm. In particular, the natural logarithm:

Properties of the Natural Logarithm

For A and B positive numbers:

1. $\ln(e^x) = x$ for any x
2. $e^{\ln x} = x$ for x positive
3. $\ln(A \cdot B) = \ln A + \ln B$
4. $\ln(\frac{A}{B}) = \ln A - \ln B$
5. $\ln(B^p) = p \ln B$ for any p

Solving Equations Using Logarithms

Property 5 of both logarithms is especially important and makes the logarithms useful for solving equations in which the unknown is in the exponent.

Example 6.2.1. Solve the equation:

$$3^x = 600.$$

Solution. The unknown x is in the exponent so we will apply a logarithm to both sides of the equation. Shall we use the common logarithm or the natural logarithm? Either one will do just fine and the solution will be the same. Let's use the common logarithm.

To solve the equation, we apply the logarithm to both sides of the equation. We can do it as to say that 3^x is equal to 600 is equivalent to saying that $\log(3^x)$ is equal to $\log(600)$. Taking the logarithm of both sides gives us an equivalent equation:

$$\log(3^x) = \log(600).$$

Note: We took the logarithm of each side. We **did NOT multiply** each side by something called "log". "log" is the name of a function and not a number. Similarly as we cannot *multiply* both sides of an equation by the radical " $\sqrt{\ }$ ". We can, however, *take the radical* of both sides.

Using Property 5 of the common logarithm, we can now take the exponent out of the logarithm and obtain:

$$x \log(3) = \log(600).$$

Note that $\log(3)$ and $\log(600)$ are constants. You can calculate their values using your calculator. So our equation has become very simple: x times a constant is equal to a constant. We divide both sides of the equation by log(3) and obtain the exact answer:

$$x = \frac{\log(600)}{\log(3)}.$$

In any real-life situation, we would want a decimal version of the answer. Using a calculator and rounding off to the third decimal place we get:

$$x = \frac{\log(600)}{\log(3)} \approx 5.823.$$

You can check for yourself that you will get the same answer if you use the natural logarithm.

Example 6.2.2. Solve for t:

$$1000 \cdot 1.05^t = 2000.$$

Solution. It will be easier if before applying the logarithm to both sides, we divide both sides by 1000:

$$1.05^t = \frac{2000}{1000}.$$

So:

$$1.05^t = 2.$$

Now we apply the logarithm to both sides:

$$\log(1.05^t) = \log(2).$$

Using Property 5:

$$t \log(1.05) = \log(2).$$

Finally, we divide both sides by the constant log(1.05) and obtain the exact solution and an approximate solution:

$$t = \frac{\log(2)}{\log(1.05)} \approx 14.207.$$

We solved the equation. Note that in doing so we found the doubling time of the function: $y = 1000 \cdot 1.05^t$.

Example 6.2.3. Solve for t:

$$500e^{0.072t} = 1800$$

Solution. We begin by dividing both sides by 500:

$$e^{0.072t} = \frac{18}{5}$$

We can use either one of the two logarithms but since the base of an exponential expression is e, it will be a bit easier to use the natural logarithm. We apply the natural logarithm to both sides and obtain an equivalent equation:

$$\ln(e^{0.072t}) = \ln(\frac{18}{5})$$

Now we can use either Property 5 of the natural logarithm with $p = 0.072t$ and get:

$$\ln(e^{0.072t}) = 0.072t \ln(e) = 0.072t \quad \text{as} \quad \ln(e) = 1.$$

Or we can use Property 1 which in essence says that for any number or expression $\heartsuit$:

$$\ln(e^{\heartsuit}) = \heartsuit.$$

Either way, we obtain:

$$\ln(e^{0.072t}) = 0.072t$$

and so

$$0.072t = \ln(\frac{18}{5})$$

We divide both sides by 0.072 and get the exact and an approximate solution to our equation:

$$t = \frac{\ln(\frac{18}{5})}{0.072} \approx 17.79$$

Example 6.2.4. Solve for x. Give the exact answer as well as its decimal approximation.

$$3 \cdot 8^x = 5 \cdot 6^x.$$

Solution. The unknown x appears in two exponential expressions. We can easily combine those expressions into one. Divide both sides of the equation by 6^x:

$$3 \cdot \frac{8^x}{6^x} = 5.$$

Divide both sides by 3:

$$\frac{8^x}{6^x} = \frac{5}{3}$$

Now remember that the power of a quotient is the quotient of the corresponding powers: $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$. Hence, the equation becomes:

$$\left(\frac{8}{6}\right)^x = \frac{5}{3}$$

Now we need a logarithm and we can use the common logarithm or the natural logarithm equally well. We shall use the common logarithm. We apply the logarithm to both sides:

$$\log\left(\left(\frac{8}{6}\right)^x\right) = \log\left(\frac{5}{3}\right)$$

By Property 5 we obtain:

$$x \log\left(\frac{8}{6}\right) = \log\left(\frac{5}{3}\right)$$

We divide both sides by the constant $\log\left(\frac{8}{6}\right)$ and obtain the exact answer:

$$x = \frac{\log\left(\frac{5}{3}\right)}{\log\left(\frac{8}{6}\right)}$$

We calculate a decimal approximation using our calculators and rounding off to three decimal places:

$$x \approx 1.776.$$

A question: What would happen if we didn't simplify the equation $3 \cdot 8^x = 5 \cdot 6^x$ before applying the logarithm to both sides? Nothing much except that calculations would become a bit more complicated. Let's try to apply the logarithm without simplifying:

$$\log(3 \cdot 8^x) = \log(5 \cdot 6^x)$$

We have to use Property 3 and expand the logarithms of the products in both sides:

$$\log(3) + \log(8^x) = \log(5) + \log(6^x)$$

Using Property 5 we get:

$$\log(3) + x\log(8) = \log(5) + x\log(6)$$

Notice that $\log(3)$, $\log(5)$, $\log(6)$, $\log(8)$ are all constants. So we group terms with x on one side, constants on the other side and obtain:

$$x(\log(8) - \log(6)) = \log(5) - \log(3)$$

and the exact solution is:

$$x = \frac{\log(5) - \log(3)}{\log(8) - \log(6)} = \frac{\log\left(\frac{5}{3}\right)}{\log\left(\frac{8}{6}\right)}.$$

Indeed, the difference of logarithms correspond to the logarithm of a quotient by Property 4. Hence, we obtained the same answer as before but the solution was definitely more complicated.

Practicing Algebraic Properties of Logarithms

Example 6.2.5. Rewrite the following expressions in terms of $\log x$ and $\log y$, or state that this is not possible.

(a) $\log\left(\dfrac{5y}{x}\right)$ (b) $\log(x^2 - 5y)$ (c) $\ln\left(7x^3y\right)$

Solution. For (a) we use the property number four from the list: the logarithm of a quotient is the difference of logarithms:

$$\log\left(\frac{5y}{x}\right) = \log(5y) - \log(x)$$

The logarithm of a product is the sum of logarithms, so:

$$\log\left(\frac{5y}{x}\right) = \log(5y) - \log(x) = \log(5) + \log(y) - \log(x).$$

(b) Before we can do anything else, we have to deal with the logarithm of a difference. **There is no formula for the logarithm of a sum or a difference!** We cannot simplify the expression in (b).

(c) Using the properties listed above, we get:

$$\ln\left(7x^3y\right) = \ln(7) + \ln(x^3) + \ln(y) = \ln(7) + 3\ln(x) + \ln(y).$$

Example 6.2.6. Fold the following expression into one logarithm:

$$3\ln y + \frac{1}{2}\ln(x) - \ln z$$

Solution. Using properties of the natural logarithm we obtain:

$$3\ln y + \frac{1}{2}\ln(x) - \ln z = \ln(y^3) + \ln(x^{\frac{1}{2}}) + \ln(z^{-1}) = \ln\left(\frac{y^3\sqrt{x}}{z}\right).$$

The Change of Base Formula for Logarithms

Even though your calculator carries only the common logarithm and the natural logarithm, you can calculate the value of any logarithm $\log_b x$, to any base b, thanks to the following change of base formulas:

$$\log_b x = \frac{\ln x}{\ln b}$$

and

$$\log_b x = \frac{\log x}{\log b}$$

Example 6.2.7. Use your calculator to find $\log_2 5$.

Solution. We can use either one of the change of base formulas. Let's use the first one and convert to the natural logarithm. We have:

$$\log_2 5 = \frac{\ln 5}{\ln 2} \approx 2.322.$$

We rounded off to three decimal places.

Of course the answer is the same if we convert to the common logarithm:

$$\log_2 5 = \frac{\log 5}{\log 2} \approx 2.322.$$

Again, we rounded off to three decimal places.

Practice Problems for Section 6.2

In Problems 1-8, use properties of logarithms to expand a given expression as much as possible and write it in terms of $\log(x)$, $\log(y)$ and $\log(z)$. If it is not possible, say so. x, y, and z are assumed to be positive.

1. $\log\left(\frac{5z}{x}\right)$

2. $\log(x^5y^3z^2)$

3. $\log\left(\frac{yz}{6x}\right)$

4. $\log\left(\frac{y^2z^3}{\sqrt{4x}}\right)$

5. $\log\left(\frac{y^2z}{x} - x^2\right)$

6. $\log(x^2 + y^2)$

7. $\log(100y\sqrt{x})$

8. $\log\left(\frac{10x}{zy}\right)$

In Problems 9-16, use properties of logarithms to expand a given expression as much as possible and write it in terms of $\ln(x)$, $\ln(y)$ and $\ln(z)$. If it is not possible, say so. x, y, and z are assumed to be positive.

9. $\ln\left(\frac{z^2}{2x}\right)$

10. $\ln(\sqrt{(x^3y^3z)})$

11. $\ln(x + y^2)$

12. $\ln\left(\frac{exy}{3z}\right)$

13. $\ln(e^2yz^{-3})$

14. $\ln\left(\frac{x}{3e}\right)$

15. $\ln\left(\frac{1}{x^3y}\right)$

16. $\ln\left(\frac{\sqrt{e}}{xz}\right)$

In Problems 17-22, fold a given expression and write it as one logarithm if possible. If it is not possible, say so. A, B, and C are assumed to be positive.

17. $\frac{\log(A)}{\log(B)}$

18. $2\log(A) - 3\log(B)$

19. $-3\ln(B) + \frac{1}{2}\ln(AB)$

20. $\log(A) \cdot \log(B)$

21. $\frac{1}{3}\ln(A) + 2\ln(B) - \ln(10)$

22. $\log(5) \cdot \log(AB)$

In Problems 23-28, solve a given equation for x or for t. Give the exact answer as well as its approximation rounded off to three decimal places.

23. $5^x = 500$

24. $1200 \cdot 1.15^t = 2400$

25. $250 \cdot 0.81^t = 125$

26. $3 \cdot 7^x = 4 \cdot 5^x$

27. $90(0.5)^{t/10} = 40$

28. $490e^{-0.098t} = 100$

In Problems 29-30, rewrite a given logarithm in terms of the natural logarithm and calculate its value using your calculator. Round off to three decimal places.

29. $\log_5(8)$

30. $\log_3(12)$

6.3 Logarithms in Applications

Logarithms appear in many applied problems. They are indispensable in the context of exponential growth and decay.

Converting Between $Q = A \cdot b^t$ and $Q = A \cdot e^{kt}$

In Section 5.5 we considered exponential functions in two forms. The standard form:

$$Q(t) = A \cdot b^t$$

and the base-e form:

$$Q(t) = A \cdot e^{kt}.$$

Recall that we called k the "continuous growth rate". In Section 5.5 we learned how to convert an exponential function given in base-e form to standard form. We simply take $b = e^k$:

$$Q(t) = A \cdot e^{kt} = A \cdot (e^k)^t = A \cdot b^t.$$

To convert an exponential function from standard form, $Q(t) = A \cdot b^t$, to base-e form, we have to find k such that:

$$e^k = b.$$

If we have such a constant k, we can easily rewrite $Q(t)$ in base-e form:

$$Q(t) = A \cdot b^t = A \cdot (e^k)^t = A \cdot e^{kt}.$$

To solve the equation

$$e^k = b$$

for k, we apply the natural logarithm to both sides:

$$\ln(e^k) = \ln(b)$$

We use Property 5 of the natural logarithm and obtain:

$$k \ln(e) = \ln(b).$$

Since $\ln(e) = 1$, we get:

$$k = \ln(b)$$

Of course! We could have noticed right away that by Property 2:

$$e^{\ln(b)} = b$$

so $k = \ln(b)$ is the constant we were looking for.

Property 2 says in essence that

$$e^{\ln(\heartsuit)} = \heartsuit$$

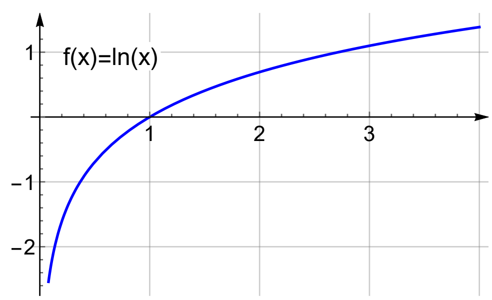
for any number or expression $\heartsuit$.

To summarize:

Converting An Exponential Function To And From Base e

- An exponential function $Q(t) = A \cdot e^{kt}$ can be rewritten in the form $Q(t) = A \cdot b^t$ for $b = e^k$.
- An exponential function $Q(t) = A \cdot b^t$ can be rewritten in the form $Q(t) = A \cdot e^{kt}$ for $k = \ln b$.
- If $b > 1$, $\ln(b) > 0$, so $k > 0$.
- If $0 < b < 1$, $\ln(b) < 0$, so $k < 0$.

The last two bullet points follow easily from the properties of the natural logarithm function $y = \ln(x)$: for inputs between 0 and 1, $\ln(x)$ is negative. For inputs greater than 1, $\ln(x)$ is positive. Recall the graph of $y = \ln(x)$:



Example 6.3.1. Rewrite the following exponential functions in terms of the natural base:

(a) $P(t) = 200 \cdot 1.089^t$ (b) $Q(t) = 1500 \cdot 0.72^t$.

Solution. (a) We know we can rewrite $P(t)$ in the form $P(t) = 200e^{kt}$ for $k = \ln 1.089 \approx 0.085$. So

$$P(t) = 200e^{0.085t}.$$

We have k positive as $b = 1.089 > 1$.

(b) $k = \ln 0.72 \approx -0.329$. k is negative as the growth factor $b = 0.72$ is less than 1. Hence, in terms of the natural base $Q(t)$ is:

$$Q(t) = 1500e^{-0.329t}$$

Note: Whenever you calculate k or the growth factor b, do not round off too crudely. Take at least three or four decimal places. A small rounding off error in k or b may lead too significant errors in the values of an exponential function.

Example 6.3.2. A common anti-anxiety medication Diazepam decays very slowly in a patient's body after a dose is taken. Suppose a patient takes a single dose of 10 mg of Diazepam. The amount $D(t)$ left after t days is given by:

$$D(t) = 10(0.74)^t.$$

(a) What is the daily percent growth rate?

(b) Rewrite $D(t)$ in the form $D(t) = 10e^{kt}$ and find the continuous daily growth rate.

Solution. (a) The daily growth rate is r such that $1 + r = 0.74$. Hence, $r = 0.74 - 1 = -0.26$. The daily growth rate is -26%. It is negative as $D(t)$ is decreasing.

(b) To find the continuous growth rate and to rewrite $D(t)$ in the form $D(t) = 10e^{kt}$, we have to find k. We know that k is the natural logarithm of the growth factor; that is:

$$k = \ln(0.74) \approx -0.301$$

So the continuous growth rate is $-0.301 = -30.1\%$. $D(t)$ in terms of the natural base is:

$$D(t) = 10e^{-0.301t}$$

Calculating the Doubling Time and the Half-Life

With logarithms we can calculate algebraically the doubling time and the halt-life of exponentially increasing or decreasing quantities.

Example 6.3.3. You deposit \$2,000 into a savings account that pays 4.5% annually. Let $B(t)$ denote your balance after t years. We have:

$$B(t) = 2000 \cdot 1.045^t.$$

(a) How long will it take for your money to double?

(b) Suppose you deposit \$5,000 instead of \$2,000. How long will it take for your money to double?

Solution. (a) To find the doubling time we have to solve the following equation for t:

$$2 \cdot 2000 = 2000 \cdot 1.045^t.$$

We begin by dividing both sides of the equation by 2000:

$$2 = 1.045^t.$$

The unknown is in the exponent, hence we apply either the common or the natural logarithm to both sides. Let's make it a habit to use the natural logarithm as that is what people do in most of applied sciences. We apply the natural logarithm to both sides and obtain:

$$\ln(2) = \ln(1.045^t)$$

which gives:

$$\ln(2) = t\ln(1.045)$$

$\ln(2)$, $\ln(1.045)$ are constants. We divide both sides by $\ln(1.045)$ and obtain:

$$t = \frac{\ln(2)}{\ln(1.045)} \approx 15.75.$$

It will take 15.75 years for your money to double.

(b) Suppose that your initial deposit is \$5,000 instead \$2,000. So $B(t)$ is now: $B(t) = 5000 \cdot 1.045^t$. How long will it take for your money to double? We have to solve the equation:

$$2 \cdot 5000 = 5000 \cdot 1.045^t.$$

We begin by dividing both sides of the equation by 5000 and obtain:

$$2 = 1.045^t.$$

This is the same equation as in the second step in part (a) so the solution is again:

$$t = \frac{\ln(2)}{\ln(1.045)} \approx 15.75.$$

The doubling time does not depend on the initial amount. It depends only on the growth factor. We don't need to know the specific value of the initial amount to find the doubling time of an exponential function.

Example 6.3.4. As we saw in Example 6.3.2, the amount of Diazepam, $D(t)$, in mg, left in a patient's system t days after a 10 mg dose is:

$$D(t) = 10e^{-0.301t}.$$

(a) Find the half-life of Diazepam.

(b) How long will it take for the initial amount of Diazepam to be reduced to 3 mg?

Solution. To find the half-life, we have to solve for t the equation:

$$10e^{-0.301t} = \frac{1}{2} \cdot 10.$$

We didn't multiply $\frac{1}{2} \cdot 10$ as the first step to solving the equation is dividing both sides by 10:

$$e^{-0.301t} = \frac{1}{2}.$$

At this point, you may notice that we would obtain the same equation regardless of the initial amount. Hence, just as the doubling time, the half-life doesn't depend on the initial amount. We apply the natural logarithm to both sides of the equation:

$$\ln(e^{-0.301t}) = \ln(\frac{1}{2}).$$

Using Property 1 or Property 5 of the natural logarithm, we get:

$$-0.301t = \ln(\frac{1}{2}).$$

We divide both sides by -0.301 which gives us the exact and an approximate answer:

$$t = \frac{\ln(\frac{1}{2})}{-0.301} \approx 2.3.$$

The half-life of Diazepam is 2.3 days. It will take 2.3 days for the half of the initial dose to decompose.

(b) We have to find t for which $D(t) = 3$, so we have to solve the equation:

$$10e^{-0.301t} = 3.$$

We divide both sides by 10, apply the natural logarithm to both sides, and use properties of the natural logarithm to obtain:

$$e^{-0.301t} = \frac{3}{10}$$

$$\ln(e^{-0.301t}) = \ln(\frac{3}{10})$$

$$-0.301t = \ln(\frac{3}{10}).$$

We divide both sides by -0.301 and get the final answer:

$$t = \frac{\ln(\frac{3}{10})}{-0.301} \approx 3.9999 \approx 4.$$

It will take about 4 days for 3 mg to be left in a patient's system from the initial dose of 10 mg.

Does this answer make sense or did we made a mistake? Let's see: the half-life is 2.3 days. Hence, 10 mg will be reduced to 5 mg after 2.3 days. This remaining 5 mg will be reduced to 2.5 mg after the next 2.3 days. So we have 5 mg left after 2.3 days and 2.5 mg left after 4.6 days. Yes, it seems reasonable that 3 mg is left after 4 days.

You can, of course, check your answer by substituting $t = 4$ into $D(t)$:

$$D(4) = 10e^{-0.301 \cdot 4} = 2.9999 \approx 3.$$

Yes, we have 3 mg left after 4 days.

The answer in (b) clearly depends on the initial amount 10 mg. If the initial dose is, say, 100 mg, it will take much longer before the amount left is 3 mg.

Example 6.3.5. During the 1986 Chernobyl disaster, radioactive Strontium-90 was released into the atmosphere[1] . Strontium-90 contaminated the nearby region and accumulated in people's

[1] https://semspub.epa.gov/work/HQ/175430.pdf, accessed: 5/24/2020

bones. (The isotope is often referred to as a "bone seeker".) The half-life of Strontium-90 is 29 years.

(a) Find the percentage of the original amount of Strontium-90 absorbed that is still left in people's bones in 2020.

(b) How many years will it take for 10% of the original amount of Strontium-90 to be left?

Solution. (a) Let $S(t)$ be the amount of Strontium-90 remaining in an affected person's bones t years after the disaster. Since Strontium-90 has a half-life, it is decaying exponentially. (As do all radioactive elements.) Denote by S_0 the initial amount absorbed. Let's find a formula for $S(t)$ in terms of the natural base:

$$S(t) = S_0 e^{kt}.$$

To find k, we use the half-life and set up an equation for k. At $t = 29$, half of S_0 is left; that is, $S(29) = \frac{1}{2} \cdot S_0$:

$$S_0 e^{k \cdot 29} = \frac{1}{2} \cdot S_0.$$

We divide both sides by S_0. The equation becomes:

$$e^{k \cdot 29} = \frac{1}{2}$$

We take the natural logarithm of both sides of the equation and simplify using the properties of the natural logarithm:

$$\ln(e^{k \cdot 29}) = \ln(\frac{1}{2})$$

$$k \cdot 29 = \ln(\frac{1}{2})$$

$$k = \frac{\ln(\frac{1}{2})}{29} \approx -0.0239.$$

We have k so we can write a formula for $S(t)$:

$$S(t) = S_0 e^{-0.0239t}.$$

The year 2020 corresponds to $t = 34$. We have:

$$S(34) = S_0 e^{-0.0239 \cdot 34} = S_0 \cdot 0.4437$$

$$\frac{S_0 \cdot 0.4437}{S_0} = 0.4437 = 44.37\%$$

In 2020, about 44% of the original amount absorbed remains in people's bones. Note that the percentage of Strontium-90 left does not depend on the initial amount S_0.

(b) We are looking for t such that $S(t) = 0.10 \cdot S_0$. That is:

$$S_0 e^{-0.0239t} = 0.10 \cdot S_0$$

We divide both sides by S_0, apply the natural logarithms and use properties of the logarithm to obtain:

$$t = \frac{\ln(0.10)}{-0.0239} \approx 96.34$$

It will take nearly 100 years of the 10% of the original amount to be left.

The Acidity of a Liquid

In chemistry the acidity of a liquid is measured on the pH scale[2]. The acidity depends on the hydrogen ion concentration in the liquid, denoted by $[\mathrm{H}^+]$ and measured in $\dfrac{\mathrm{mol}}{\mathrm{L}}$ — moles per liter. The pH is defined as:

$$\mathrm{pH} = -\log[\mathrm{H}^+]$$

(The logarithm in the formula is the common logarithm.)

Example 6.3.6. The hydrogen ion concentration of lemon juice is $[\mathrm{H}^+] = 0.01\ \dfrac{\mathrm{mol}}{\mathrm{L}}$. Find the pH of lemon juice.

Solution. According to the pH formula, we have:

$$\mathrm{pH} = -\log(0.01) = -\log(10^{-2}) = -(-2) = 2.$$

The pH of lemon juice is 2.

Example 6.3.7. The pH of gastric acid is 1. Find the hydrogen ion concentration $[\mathrm{H}^+]$ in gastric acid.

Solution. We have:

$$1 = -\log[\mathrm{H}^+]$$

Hence:

$$\log[\mathrm{H}^+] = -1$$

By the definition of the common logarithm:

$$10^{-1} = [\mathrm{H}^+]$$

Thus, $[\mathrm{H}^+] = 0.1\ \dfrac{\mathrm{mol}}{\mathrm{L}}$.

Practice Problems for Section 6.3

In Problems 1-4, rewrite a given exponential function $y = A \cdot b^t$ in terms of the natural base; that is, in the form $y = A \cdot e^{kt}$. Round off k to four decimal places.

1. $y = 208(0.85)^t$
2. $y = (1.09)^t$
3. $y = 700(1.32)^t$
4. $y = 2000(0.91)^t$

5. If a bank offers an interest rate of 6.1% with *interest compounded continuously* (rather than once a year), your balance $B(t)$ after t years is:

$$B(t) = B_0 e^{0.061t}$$

[2] http://chemistry.elmhurst.edu/vchembook/184ph.html, accessed: 5/25/2020

where B_0 is your initial deposit, in dollars.

(a) How long will it take for your money to double?

(b) How long will it take for your money to triple?

6. A laboratory culture of *Salmonella enterica* starts with S_0 bacteria at $t = 0$ and doubles every 30 minutes.[3] Let $S(t)$ be the number of bacteria in the culture at time t, in minutes.

   (a) Write a formula for $S(t)$ in terms of the natural base e.

   (b) How long will it take for $S(t)$ to triple? Does the tripling time depend on the initial amount S_0?

7. Bacterial population dynamics is not as simple as measuring the doubling time of a bacterium in the laboratory, under optimal growth conditions. A more challenging question is to find the doubling time of a bacterium in its natural environment, for example, in the gut. The doubling time of *Salmonella enterica* in the gut is 25 hours[4]. Let G_0 be an initial amount of *Salmonella enterica* in a patient's gut at $t = 0$. Let $G(t)$ be the amount t hours later.

   (a) Find a formula for $G(t)$ in base-e form.

   (b) How long will it take for the bacteria to reach 150% of the initial amount?

8. Let $N = N(t)$ be the amount of nicotine, in milligrams, in the bloodstream of a person after a cigarette is smoked. Time t is measured in hours. As with most drugs, the process of elimination of nicotine from the body is a process of exponential decay. The half-life of nicotine is 2 hours and the amount of nicotine absorbed from a cigarette is 2 mg.

   (a) Find a formula in the form $N(t) = A \cdot e^{kt}$ for the function $N(t)$.

   (b) How long will it take for the initial amount of nicotine to be reduced to 0.7 mg?

9. Let $C(t)$ be the amount of caffeine that remains in the person's body t hours after finishing a cup of coffee. The amount of caffeine absorbed from a cup of coffee is 96 mg. The continuous growth rate of caffeine in the body is -13.86%.

   (a) Find a formula for $C(t)$ in terms of the natural base.

   (b) Find the half-life of caffeine in the body. Round off your answer to two decimal places. Include units with your answer.

   (c) How long will it take for $C(t)$ to be reduced to 30% of the initial amount? Does the answer depend on the initial amount?

   (d) How long will it take for $C(t)$ to be reduced to 10 mg? Does the answer depend on the initial amount?

10. The value of an antique chair, $V(t)$, t years after it was purchased for \$7000 increases by 15% per year.

   (a) Write a formula for $V(t)$ in the form $V(t) = A \cdot b^t$.

   (b) When will the value reach \$16000?

[3]https://www.ncbi.nlm.nih.gov/pmc/articles/PMC6015860/, accessed: 6/26/20

[4]https://www.ncbi.nlm.nih.gov/pmc/articles/PMC6015860/, accessed: 6/26/20

11. The pH of orange juice is 3. Find the hydrogen ion concentration $[H^+]$ in orange juice.

12. The hydrogen ion concentration $[H^+]$ in tomato juice is 0.0001 moles per liter. Find the pH of tomato juice.

Chapter 7

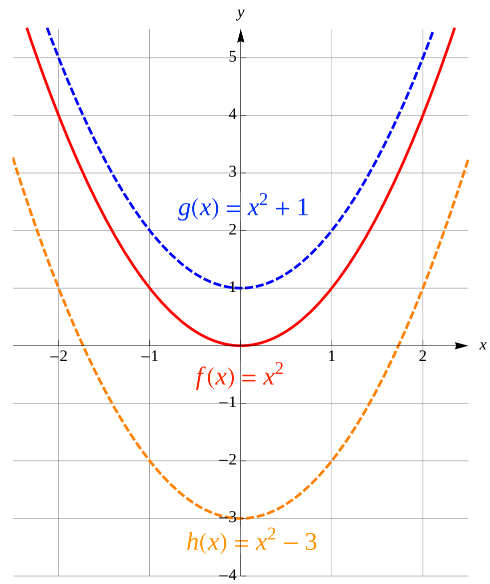
New Functions from Old

7.1 Vertical and Horizontal Shifts

Vertical Shifts

What happens if we change the function $f(x) = x^2$ by adding or subtracting a number from it? For instance, what would the graphs of $g(x) = x^2 + 1$ and $h(x) = x^2 - 3$ look like? In this case, the graph of $g(x) = x^2 + 1$ is the graph of $f(x) = x^2$ *shifted up one unit* and the graph of $h(x) = x^2 - 3$ is the graph of $f(x) = x^2$ *shifted down 3 units*, as can be seen to right.

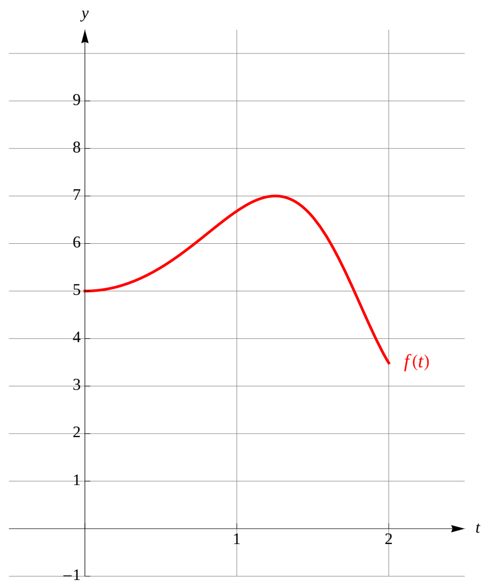
In general, if we begin with the graph of a function $y = f(x)$, then the function $g(x) = f(x) + k$ can be given by the formula $g(x) = y + k$. That is, the y-values of $g(x)$ are the y-values of $f(x)$ adjusted by the addition of the number k. Hence the graph of $g(x) = f(x) + k$ is the graph of $f(x)$ shifted either upward or downward depending upon the value of k.



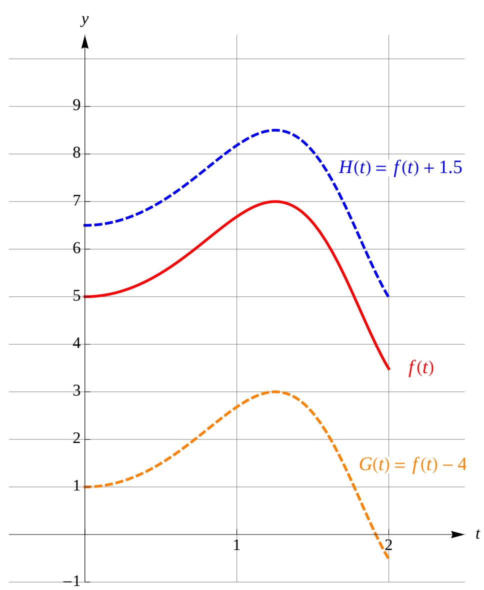
graphs of $f(x) = x^2$, $g(x) = x^2 + 1$, and $h(x) = x^2 - 3$

> **Vertical Shifts**
>
> Given a function $f(x)$ and a constant k, $g(x) = f(x) + k$ is a *vertical shift* of the graph of $f(x)$. If k is positive, then:
>
> - the graph of $g(x) = f(x) + k$ is the graph of $f(x)$ shifted **UP** k units.
> - the graph of $g(x) = f(x) - k$ is the graph of $f(x)$ shifted **DOWN** k units.

Example 7.1.1. The graph of a function $f(t)$ is given below. Plot $G(t) = f(t) - 4$ and $H(t) = f(t) + 1.5$.



Solution. The graph of $G(t) = f(t) - 4$ is the graph of $f(t)$ shifted down 4 units while the graph of $H(t) = f(t) + 1.5$ is the graph of $f(t)$ shifted up 1.5 units. Hence we have the following.



Example 7.1.2. The total monthly cost C to operate a company depends upon the fixed costs for facility rental and equipment as well as the number of units, p, that the company produces that month. The table below shows this relationship for a typical month.

p	0	10	20	30
C	2500	2650	2800	2950

Suppose the rent on the facility went up by \$200 per month. Create a table for the resulting total monthly cost function S as a function of the number of units produced, p.

Solution. The outputs in the original table give the original total monthly cost as a function of the number of units produced, p. If the rent were to go up by \$200 per month, then the resulting total monthly cost function would be $S = C + 200$. This is a vertical shift of the original cost function C, and values for it can be obtained by adding 200 to each of the outputs in the original table, as can be seen below.

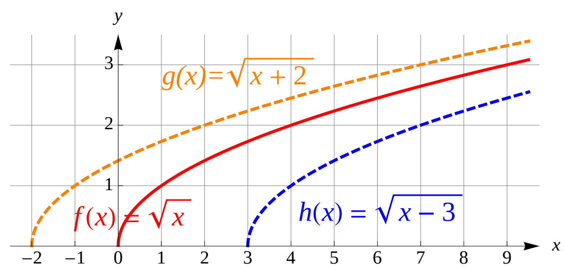
p	0	10	20	30
S	2700	2850	3000	3150

Horizontal Shifts

We saw that adding or subtracting a number from the output of a function results in a vertical shift. What happens if we add or subtract a number from the *input* of a function instead?

For instance, we know that $y = \sqrt{x} + 2$ would result in shifting the function $f(x) = \sqrt{x}$ up two units. But what if we added or subtracted a number *under* the square root instead? That is, what if we added or subtracted a quantity from the *input* of the original function, rather than the output?

To illustrate, we will consider the functions $g(x) = \sqrt{x+2}$ and $h(x) = \sqrt{x-3}$. The graph of $g(x) = \sqrt{x+2}$ is the graph of $f(x) = \sqrt{x}$ shifted *left* two units while $h(x) = \sqrt{x-3}$ is the graph of $f(x) = \sqrt{x}$ shifted *right* 3 units, as can be seen to the right.



graphs of $f(x) = \sqrt{x}$, $g(x) = \sqrt{x+2}$, and $h(x) = \sqrt{x-3}$.

To help explain why this is the case, notice that the formula $f(x) = \sqrt{x} = y$ is solved for y and not x. We can change this by squaring both sides of $\sqrt{x} = y$:

$$x = y^2.$$

If we were then to solve the formula $g(x) = \sqrt{x+3} = y$ for x, we would square both sides of $\sqrt{x+2} = y$ and then subtract 2:

$$\begin{aligned} x + 2 &= y^2 \\ x + 2 - 2 &= y^2 - 2 \\ x &= y^2 - 2. \end{aligned}$$

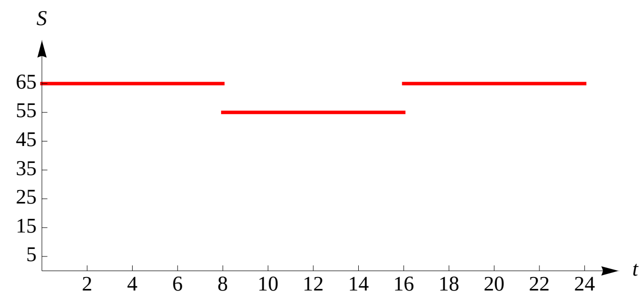
Notice that the formula for the x-values of $g(x) = \sqrt{x+2}$ involves *subtracting* 2 from the formula for the x-values of the original function $f(x) = \sqrt{x}$. This means that for each specific y-value, the corresponding x-value of $g(x) = \sqrt{x+2}$ is two units in the *negative direction* from that of the original function $f(x) = \sqrt{x}$; that is, two units to the left.

Horizontal Shifts

Given a function $f(x)$ and a constant h, $g(x) = f(x+h)$ is a *horizontal shift* of the graph of $f(x)$. If h is positive, then

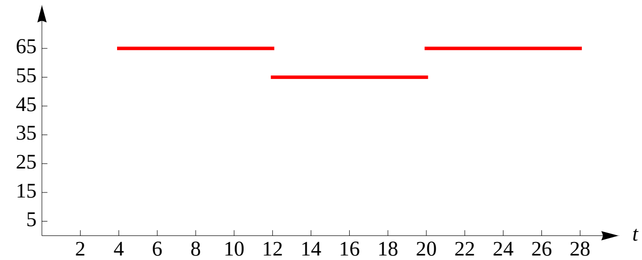
- the graph of $g(x) = f(x+h)$ is the graph of $f(x)$ shifted **LEFT** h units.
- the graph of $g(x) = f(x-h)$ is the graph of $f(x)$ shifted **RIGHT** h units.

Example 7.1.3. In an effort to save on heating costs, the thermostat in a professor's home has been programmed to keep his house at 65 degrees while he is home from 4pm to 8am the following day and to drop the temperature to 55 degrees while he is at work from 8am to 4pm each day. This is illustrated in the graph below, where t corresponds to hours past since midnight and S corresponds to the thermostat setting.



Due to a schedule change requiring him to teach a night class, the professor needs to shift his entire work schedule later by four hours and hence adjust his thermostat setting accordingly. Sketch a graph reflecting this change.

Solution. Shifting his entire work schedule forward by 4 hours is equivalent to moving it forward in time by 4 hours. This corresponds to a horizontal shift to the *right* 4 units. This means that we must graph the corresponding horizontal shift $S(t-4)$.



Note that the original graph shows a thermostat setting of 55 degrees between times $t = 8$ and $t = 16$, corresponding to a work day stretching from 8am to 4pm. The shifted graph shows a thermostat setting of 55 degrees between times $t = 12$ and $t = 20$, corresponding to a work day stretching from 12pm to 8pm (pushed back 4 hours, as it was intended).

Example 7.1.4. Consider the function $f(x)$ given in the table below. Construct a table of values for $g(x) = f(x + 6.2)$.

x	0	5	10	15
$f(x)$	6	-16.5	7.3	10.4

Solution. The function $g(x) = f(x + 6.2)$ is a horizontal shift of $f(x)$; its graph would be the graph of $f(x)$ shifted to the left 6.2 units. This means that to construct a table for $g(x)$, we should take the table for $f(x)$ and subtract 6.2 from each of the inputs while leaving the outputs unchanged.

x	-6.2	-1.2	3.8	8.8
$g(x)$	6	-16.5	7.3	10.4

Combining Vertical and Horizontal Shifts

Shifting vertically and shifting horizontally are two illustrations of *transformations* of functions; they take an original function and transform (or change) it by shifting it in some way. Multiple transformations may be applied to an original function; for instance, both a vertical and a horizontal shift can be applied to an original function: The graph of the function $y = (x + 2)^2 - 7$ is obtained by shifting the graph of $f(x) = x^2$ down 7 units and left 2 units.

Example 7.1.5. For each of the following, identify the function being transformed and describe the transformations being applied to it.

(a) $y = (x - 10)^5 + 4$

(b) $y = 2^{t+1} + 3$

Solution. (a) The original function is $f(x) = x^5$. Since 4 is being added on the end, the vertical shift is up 4 units. Since 10 is being subtracted from the input variable x, the horizontal shift is right 10 units.

(b) The original function is $f(t) = 2^t$. Since 3 is being added on the end, the vertical shift is up 3 units. Since 1 is being added to the input variable t, the horizontal shift is left 1 unit.

Example 7.1.6. Write the formula for the function obtained when...

(a) the graph of $f(t) = 10(0.5)^t$ is shifted up 1 unit and to the left 4 units.

(b) the graph of $g(t) = \sqrt[3]{x}$ is shifted down 4 units and to the right 3 units.

Solution. (a) To shift up 1 unit, we add 1 to the whole function and to shift to the left 4 units we add 4 to the input variable t in the exponent. Hence the new function will have formula $y = 10(0.5)^{t+4} + 1$.

(b) To shift down 4 units, we subtract 4 from the whole function and to shift to the right 3 units we subtract 3 from the input variable x under the cube root. Hence the new function will have the formula $y = \sqrt[3]{x-3} - 4$.

Practice Problems for Section 7.1

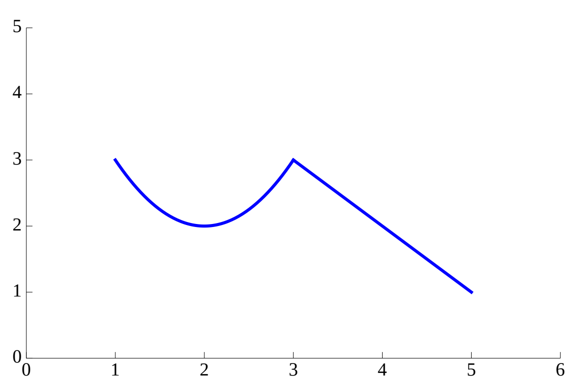
In Problems 1-8, give the formula for the function $g(x)$ satisfying each of the following.

1. The graph of $g(x)$ is the graph of $f(x)$ shifted down 7 units.
2. The graph of $g(x)$ is the graph of $f(x)$ shifted up 1.5 units.
3. The graph of $g(x)$ is the graph of $f(x)$ shifted right 2 units.
4. The graph of $g(x)$ is the graph of $f(x)$ shifted left 3 units.
5. The graph of $g(x)$ is the graph of $f(x)$ shifted up 2.3 units and shifted right 6 units.
6. The graph of $g(x)$ is the graph of $f(x)$ shifted down 1.63 units and right 0.5 units.
7. The graph of $g(x)$ is the graph of $f(x)$ shifted up 12 units and right 10 units.
8. The graph of $g(x)$ is the graph of $f(x)$ shifted down 3.2 units and left 7.1 units.

In Problems 9-12, identify the function being transformed and describe the transformations being applied to it.

9. $f(x) = (x+4)^2$
10. $g(t) = e^t - 3.2$
11. $h(w) = \sqrt{w - 4.1} + 5.2$
12. $y = (0.9)^{t+1} - 3$

In Problems 13-16, use the graph of $f(x)$ shown below to sketch the graph of each of the following transformations of $f(x)$.



13. $y = f(x) + 3$

14. $y = f(x - 2)$

15. $y = f(x + 1) - 2$

16. $y = f(x - 6) + 1.5$

In Problems 17-20, use the table of values of $f(t)$ shown below to write the table of values for each of the following transformations of $f(t)$.

t	0	5	10	15
$f(t)$	0.1	0.2	0.4	0.8

17. $y = f(t + 3)$

18. $y = f(t) - 2$

19. $y = f(t - 2) + 6$

20. $y = f(t + 3.1) + 1.1$

21. The function $H = f(t)$ gives the temperature (in degrees Fahrenheit) of the water in a spa t minutes after the spa heater has been turned on.

    (a) Write the formula for a function $g(t)$ that represents the temperature (in degrees Fahrenheit) of the water in the spa if the spa heater is turned on 15 minutes earlier.

    (b) Write a formula for the function $h(t)$ that represents the temperature (in degrees Fahrenheit) of the water in the space if the temperature of the water in the spa were 5 degrees warmer at the time that the heater was turned on.

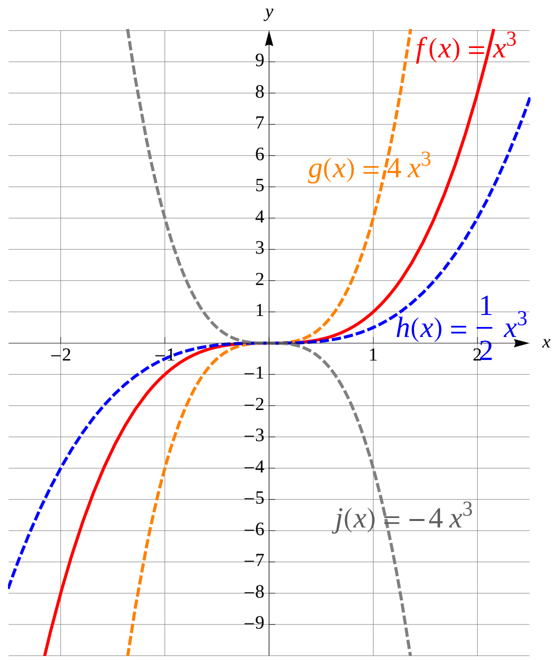
22. The function $C(t)$ gives the concentration (in nanograms per milliliter) of a certain medication in a patient's bloodstream t minutes after a dose of the medication has been administered to a patient, assuming that the patient has 0 nanograms per milliliter in their bloodstream at time $t = 0$. Write the formula for the function $g(t)$ that supposes that at time $t = 0$ the patient had 2.3 nanograms per milliliter of the medication in their bloodstream.

7.2 Vertical and Horizontal Scaling

Vertical Scaling

Consider the function $f(x) = x^3$. What would happen if we multiply this function by a nonzero number? For instance, what would the graphs of $g(x) = 4x^3$ and $h(x) = \frac{1}{2}x^3$ look like? In this case, the graph of $g(x) = 4x^3$ is the graph of $g(x) = x^3$ *stretched vertically* by multiplying each of the original outputs of $f(x) = x^3$ by 4 and the graph of $h(x) = \frac{1}{2}x^3$ is the graph of $f(x) = x^3$ *compressed vertically* by dividing each of the original outputs of $f(x) = x^3$ by 2 (equivalently, we can think of this as multiplying the original outputs by $\frac{1}{2}$).

What if we were to multiply by a negative number? The graph of $j(x) = -4x^3$ is the graph of $g(x) = x^3$ *stretched vertically* by multiplying each of the original outputs of $f(x) = x^3$ by 4, but also *reflected over the x-axis.*



graphs of $f(x) = x^3$, $g(x) = 4x^3$, $h(x) = \frac{1}{2}x^2$, and $j(x) = -4x^3$.

In general, if we begin with the graph of a function $y = f(x)$, then the function $g(x) = af(x)$ can be given by the formula $g(x) = ay$. That is, the y-values of $g(x)$ are the y-values of $f(x)$ adjusted by the multiplication of the number a. Hence the graph of $g(x) = af(x)$ is the graph of $f(x)$ either stretched (if $|a| > 1$) or compressed (if $|a| < 1$) vertically, and additionally reflected over the x-axis if a is negative.

Vertical Scaling

Given a function $f(x)$ and a constant a, $g(x) = af(x)$ is a *vertical scaling* of the graph of $f(x)$ and also a reflection of $f(x)$ over the x-axis if a is negative. In either case

- the graph of $g(x) = af(x)$ is the graph of $f(x)$ with its outputs multiplied by a.

Example 7.2.1. A table of values for a function $f(t)$ is given below. Give the table of values for $g(t)$, whose graph is that of $f(t)$ reflected over the t-axis and compressed vertically by a factor of 10.

t	0	1.5	3	4.5
$f(t)$	45	23	-31	1

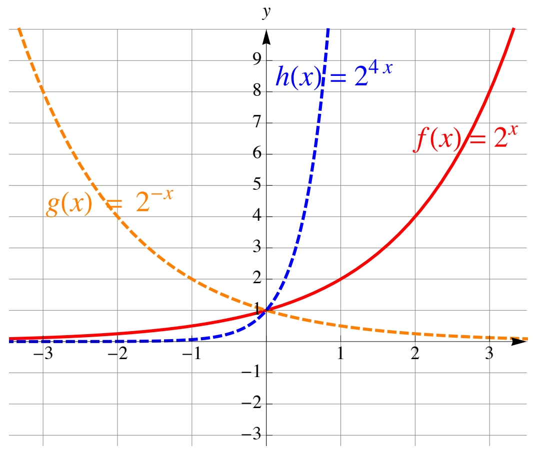
Solution. A reflection over the t-axis will change the sign of every output of $f(t)$, while vertical compression by a factor of 10 means that every output must be divided by 10. We could think of $g(t)$ as having the formula $g(t) = -\frac{1}{10}f(t)$ and the resulting table of values would be

t	0	1.5	3	4.5
$g(t)$	-4.5	-2.3	3.1	-0.1

Horizontal Scaling

Vertical scaling results from multiplying all of the outputs of a given function by the same number. We can also take an original function and multiply all of its *inputs* by the same number.

Consider $f(x) = 2^x$. The function $g(x) = 2^{-x}$ is the graph of $f(x) = 2^x$ *reflected over the* y*-axis* while $h(x) = 2^{(4x)}$ is the graph of $g(x) = 2^x$ *scaled horizontally* by dividing each original input by 4 (or, equivalently, multiplying each original input by $\frac{1}{4}$).



graphs of $f(x) = 2^x$, $g(x) = 2^{-x}$, and $h(x) = 2^{4x}$.

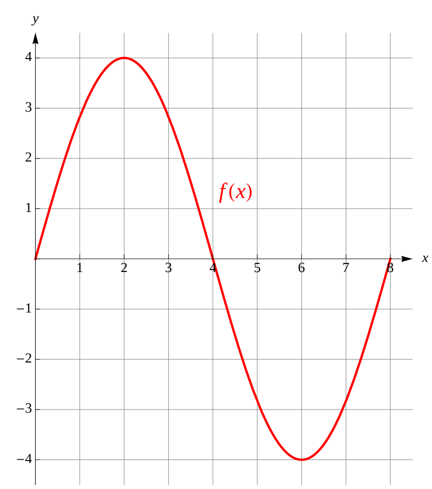
Horizontal Scaling

Given a function $f(x)$ and a constant b, $g(x) = f(bx)$ is a *horizontal scaling* of the graph of $f(x)$ and also a reflection of $f(x)$ over the y-axis if b is negative. In either case

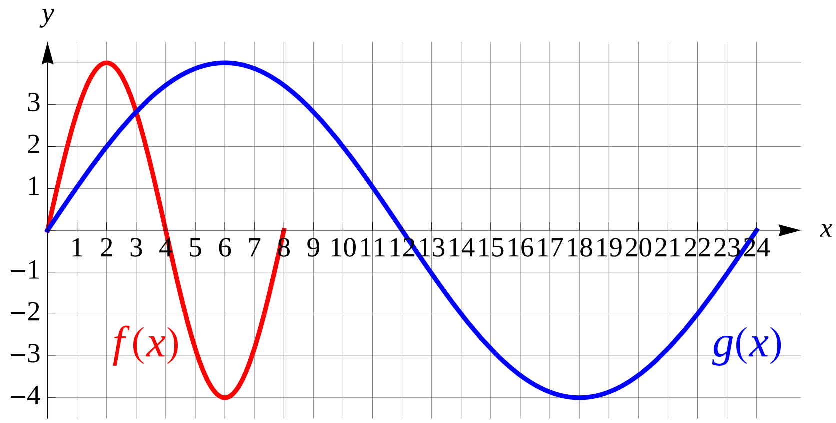
- the graph of $g(x) = f(bx)$ is the graph of $f(x)$ with its inputs multiplied by $\frac{1}{b}$.

Example 7.2.2.

The graph of a function $y = f(x)$ is shown to the right. Write the formula for the function $g(x)$ whose graph is that of $f(x)$ stretched horizontally by a factor of 3 and also graph $g(x)$.



Solution. In order to stretch $f(x)$ by a factor of 3, we need to multiply each of its inputs by 3. To accomplish this, we must let the b in the horizontal scaling formula be given by $\frac{1}{3}$. Why? The function $g(x) = f(bx)$ is the graph of $f(x)$ scaled horizontally by multiplying each of its inputs by $\frac{1}{b}$. Since we need to multiply each input by 3, we need $\frac{1}{b} = 3$, which implies that $b = \frac{1}{3}$. So $g(x) = f\left(\frac{1}{3}x\right)$. The graph of $g(x)$ is formed by taking each ordered pair on $f(x)$ and leaving its y-value the same but multiplying its x-value by 3.



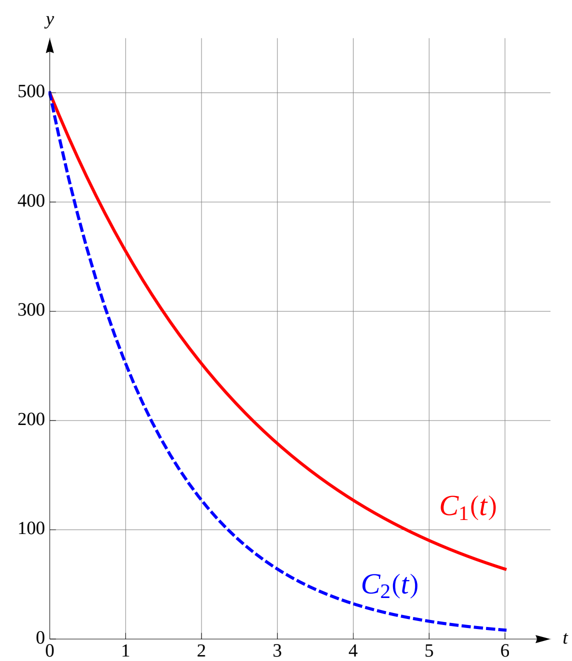
Example 7.2.3. A certain medication metabolizes so that the concentration in the body t hours after taking a 500mg dose is modeled by the function $C_1(t) = 500(0.71)^t$. A 500mg dose of a different medication follows the same pattern but metabolizes twice as fast. Use your knowledge of horizontal scaling to represent drug concentration curve C_2 of this different medication.

Solution.

In order for the second medication to metabolize twice as fast as the first while following the same pattern, the graph of C_2 must be the graph of C_1 compressed horizontally by dividing each of the inputs of C_1 by 2 (or, equivalently, multiplying them by $\frac{1}{2}$). For instance, the concentration of the first medication 6 hours after a dose should be the same as the concentration of the second medication 3 hours after a dose. This corresponds to

$$C_2(t) = C_1(2t) = 500(0.71)^{2t}.$$

It can be verified that this formula makes sense by graphing the two curves, as shown to the right.



Combining Shifts, Scaling, and Reflections

The transformations of functions introduced in this section can be combined with each other and with horizontal/vertical shifting as described in the previous section. When identifying the transformations that have been applied to an original function, we work our way from the inside of the function out by looking at what is happening to the variable x and in what order via order of operations. This is explained below.

Transformations of Functions in General

The graph of the transformation

$$g(x) = af\big(b(x-h)\big) + k$$

is the graph of $f(x)$ with the following transformations applied **in the listed order**:

1. Shifted horizontally according to h.
2. Scaled horizontally according to b with a reflection over the y-axis if b is negative.
3. Scaled vertically according to a with a reflection over the x-axis if a is negative.
4. Shifted vertically according to k.

Practice Problems for Section 7.2

In Problems 1-8, give the formula for the function $g(x)$ satisfying each of the following.

1. The graph of $g(x)$ is the graph of $f(x)$ stretched vertically by a factor of 3.
2. The graph of $g(x)$ is the graph of $f(x)$ compressed vertically by a factor of 20.
3. The graph of $g(x)$ is the graph of $f(x)$ stretched vertically by a factor of 3 and reflected over the x-axis.
4. The graph of $g(x)$ is the graph of $f(x)$ shifted to the right 2 units and compressed vertically by a factor of 4.
5. The graph of $g(x)$ is the graph of $f(x)$ reflected over the y-axis, compressed vertically by a factor of 5, and shifted up 3 units.
6. The graph of $g(x)$ is the graph of $f(x)$ stretched horizontally by a factor of 7.
7. The graph of $g(x)$ is the graph of $f(x)$ compressed horizontally by a factor of 2.

8. The graph of $g(x)$ is the graph of $f(x)$ shifted to the right 2 units, stretched horizontally by a factor of 5, reflected over the y axis, stretched vertically by a factor of 6, and shifted up 10 units.

In Problems 9-12, identify the function being transformed and describe the transformations being applied to it.

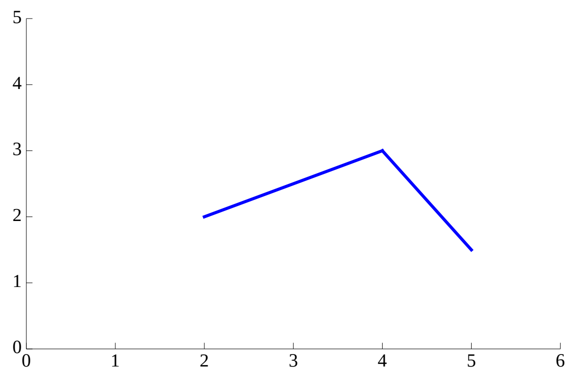
9. $f(x) = -2(x+4)^3$

10. $g(t) = \sqrt[3]{2(t-5)} + 9$

11. $h(w) = \frac{1}{4}e^{-2w} - 7$

12. $y = 2^{7(x+1)} - 3$

In Problems 13-18, use the graph of $f(x)$ shown below to sketch the graph of each of the following transformations of $f(x)$.



13. $y = 2f(x)$

14. $y = -f(x)$

15. $y = f(2x)$

16. $y = f(-3x)$

17. $y = \frac{1}{2}f(x+1) - 3$

18. $y = 3f(-(x-5)) - 8$

In Problems 19-22, use the table of values of $f(x)$ shown below to write the table of values for each of the following transformations of $f(x)$.

x	0	2	4	6
$f(x)$	-2	0.5	6	-10

19. $y = -2f(x)$

20. $y = 0.5f(x)$

21. $y = -2f(x+1) - 3$

22. $y = 3f(-2x) + 1$

7.3 Composing and Decomposing Functions

Suppose you wanted to convert 12000 inches to miles. The function

$$g(x) = \frac{1}{12}x$$

takes an input x in inches and yields an output in feet (because there are 12 inches in one foot). Similarly, the function

$$f(x) = \frac{1}{5280}x$$

takes an input x in feet and yields an output in miles (because there are 5280 feet in a mile). We could thus covert 12000 inches to miles by first converting it to feet by computing

$$g(12000) = \frac{1}{12}(12000) = 1000$$

and then taking 1000 feet and converting that to miles by calculating

$$f(1000) = \frac{1}{5280}(1000) \approx 0.189.$$

Thus we arrive at the conclusion that 12000 inches is approximately 0.189 miles.

What if we wanted a single function that could convert inches to miles, so that we only had to perform one calculation rather than two? Given two functions, it is possible to generate another function by *composing* one with the other. This can be done by substituting one function into the other.

Definition of a Composition of Functions

Given two functions f and g, we denote the **composition of f with g** by $f \circ g$ and define

$$(f \circ g)(x) = f(g(x)).$$

We read the above as "f of g of x." $(f \circ g)(x)$ is defined for inputs x such that x is in the domain of g and $g(x)$ is in the domain of f.

f is called the "outside" or the "outer" function of the composition $f(g(x))$; g is called the "inside" or the "inner" function of the composition $f(g(x))$.

Example 7.3.1. Let $f(x) = \frac{1}{5280}x$ and let $g(x) = \frac{1}{12}x$. Find $(f \circ g)(x)$.

Solution.

By definition, the function

$$(f \circ g)(x) = f(g(x)).$$

This means that we are taking the function $f(x)$ and substituting $g(x) = \frac{1}{12}x$ into it. Thus we have that

$$\begin{aligned} f(g(x)) &= f\left(\frac{1}{12}x\right) \\ &= \frac{1}{5280}\left(\frac{1}{12}x\right) \\ &= \frac{1}{63360}x. \end{aligned}$$

Observe that in the above example, $(f \circ g)(12000) = \frac{1}{63360}(12000) \approx 0.189$, so this composite function in fact is the formula for converting an input x in inches to miles.

Example 7.3.2. Let $f(x) = x^2 - 2x + 1$, $g(x) = 2 - x$, and let $h(x) = \sqrt{x-1}$. Find each of the following.

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$ (c)$(h \circ f \circ g)(x)$ (d) $(h \circ h)(x)$

Solution. (a) We have that

$$\begin{aligned} f(g(x)) &= f(2-x) \\ &= (2-x)^2 - 2(2-x) + 1 \\ &= 4 - 4x + x^2 - 4 + 2x + 1 \\ &= x^2 - 2x + 1. \end{aligned}$$

(b) We have that

$$\begin{aligned} g(f(x)) &= g(x^2 - 2x + 1) \\ &= 2 - (x^2 - 2x + 1) \\ &= 2 - x^2 + 2x - 1 \\ &= -x^2 + 2x + 1. \end{aligned}$$

(c) We have that

$$\begin{aligned} h(f(g(x))) &= h(x^2 - 2x + 1) \text{ since we know } f(g(x)) = x^2 - 2x + 1 \text{ from part (a)} \\ &= \sqrt{x^2 - 2x + 1 - 1} \\ &= \sqrt{x^2 - 2x}. \end{aligned}$$

(d) We have that

$$\begin{aligned} h(h(x)) &= h(\sqrt{x-1}) \\ &= \sqrt{(\sqrt{x-1}) - 1} \\ &= \sqrt{\sqrt{x-1} - 1}. \end{aligned}$$

Note: Notice that the two functions $(f \circ g)(x) = f(g(x))$ and $(g \circ f)(x) = g(f(x))$ in the above example are different. **The order in which you compose functions matters.**

To compose functions is important but it is even more important to be able to *decompose* a complicated function into simpler functions; that is, represent a more complicated function as a composition of simpler functions.

Example 7.3.3. Express each of the following as a composition of two functions $f(x)$ and $g(x)$ so that $y = (f \circ g)(x)$.

(a) $y = (3x - 10)^9$ (b) $y = \sqrt{x+1}$ (c) $y = \ln(x^2 + 2)$

Solution. (a) To represent the function $y = (3x - 10)^9$ as a composition $y = f(g(x))$, think of the function as a sequence of operations. To obtain output y from an input x we perform the following operations: multiply by 3, subtract 10, and then take the power 9 of the result. To decompose the function, split this sequence of operations into two simpler chunks. "Multiply by 3 and subtract 10" — that is the first group of operations which gives us our inner function $g(x) = 3x - 10$. The next operation is "take power 9" which gives us the outer function $f(x) = x^9$.

Does this work? Let us compute $(f \circ g)(x) = f(g(x))$ for this choice of $f(x)$ and $g(x)$:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(3x - 10) \\ &= (3x - 10)^9.\end{aligned}$$

This is equivalent to our original function y, so we have chosen $f(x)$ and $g(x)$ appropriately, and we are done.

(b) The sequence of operations performed on the input to get the corresponding output is: add 1 and then take the square root of the result. Take the inner function g which adds 1 and the outer function f which takes the square root. That is, take $g(x) = x + 1$ and $f(x) = \sqrt{x}$. The given function $y = \sqrt{x+1}$ is the composition $f(g(x))$:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x + 1) \\ &= \sqrt{x+1}.\end{aligned}$$

(c) The function represents the sequence of operations: take the square of the input, add 2, and then take the natural logarithm of the result. Hence, we can write the function as $f(g(x))$ for $g(x) = x^2 + 2$ and $f(x) = \ln(x)$. Of course:

$$\begin{aligned}f(g(x)) &= \\ &= f(x^2 + 2) \\ &= \ln(x^2 + 2)\end{aligned}$$

which gives our original function $y = \ln(x^2 + 2)$.

Example 7.3.4. Decompose the function

$$h(x) = \frac{1}{\sqrt{x+1}}$$

into two simpler functions f and g.

Solution. The sequence of operations that h performs on an input to obtain the output is: add 1, take the square root, take the reciprocal of the result. Let's group them as follows: "add 1 and take the square root" for the inner function and "take the reciprocal" for the outer function. In other words we take the inner function $g(x) = \sqrt{x+1}$ and the outer function $f(x) = \dfrac{1}{x}$. It works as:

$$\begin{aligned} h(x) &= f(g(x)) \\ &= f(\sqrt{x+1}) \\ &= \frac{1}{\sqrt{x+1}} \end{aligned}$$

Is it the only way to decompose h? No. Usually there are many ways to decompose a given function. For the function h given above, take $g(x) = x + 1$ and $f(x) = \frac{1}{\sqrt{x}}$. Those two functions work as well:

$$\begin{aligned} h(x) &= f(g(x)) \\ &= f(x+1) \\ &= \frac{1}{\sqrt{x+1}} \end{aligned}$$

Example 7.3.5. Decompose the function

$$m(x) = e^{x^3-2}$$

into two simpler functions f and g.

Solution. The sequence of operations for $m(x)$ is: take the cube, subtract 2 and then take the natural exponential of the result. The following functions should work: $g(x) = x^3 - 2$, $f(x) = e^x$. Of course:

$$\begin{aligned} m(x) &= f(g(x)) \\ &= f(x^3-2) \\ &= e^{x^3-2} \end{aligned}$$

Practice Problems for Section 7.3

In Problems 1-10, find $(f \circ g)(x)$ and $(g \circ f)(x)$ for the functions $f(x)$ and $g(x)$ given.

1. $f(x) = x + 1$ and $g(x) = 2x$
2. $f(x) = 3x - 1$ and $g(x) = 5 - \frac{1}{2}x$
3. $f(x) = 5x - 10$ and $g(x) = 3x^2 - 5$
4. $f(x) = x^2 - 6x + 1$ and $g(x) = 8x + 1$

5. $f(x) = \sqrt{x}$ and $g(x) = 4 - x$

6. $f(x) = \sqrt{x-7}$ and $g(x) = \sqrt{x+7}$

7. $f(x) = \frac{2x+1}{5}$ and $g(x) = 2e^x$

8. $f(x) = 5^x$ and $g(x) = 2x - 1$

9. $f(x) = \dfrac{x}{x^2 - 9}$ and $g(x) = 3x + 1$

10. $f(x) = \dfrac{2x-3}{x+1}$ and $g(x) = \dfrac{1}{2x-1}$

In Problems 11-14, find $(f \circ g)(2)$ for the functions $f(x)$ and $g(x)$ given.

11. $f(x) = 6x - 1$ and $g(x) = 2x$

12. $f(x) = 2x - 1$ and $g(x) = \sqrt{8x}$

13. $f(x) = \frac{3x+1}{2}$ and $g(x) = \frac{1}{x}$

14. $f(x) = \sqrt{x}$ and $g(x) = 2x^2 - 2x$

In Problems 15-20, express each of the following as a composition of two functions $f(x)$ and $g(x)$ so that $y = (f \circ g)(x)$.

15. $y = 2(x-5)^5 + 1$

16. $y = \sqrt[3]{7-x}$

17. $y = 2^{\sqrt{x}}$

18. $y = \dfrac{1}{5x-7}$

19. $y = \ln(16x + 5x^2)$

20. $y = \left(\dfrac{2x+1}{x-1}\right)^3$

21. $y = e^{3x+2}$

22. $y = e^{\sqrt{x-5}}$

7.4 Inverse Functions

In 1987, Amos Dolbear published the article “The Cricket as a Thermometer.” This paper included the formulation of what is now known as Dolbear’s Law: a formula that states the relationship between the air temperature and the number of times that a cricket chirps in a given span of time. This law states that the air temperature T in degrees Fahrenheit is a function of the number of times N that a cricket chirps in 15 seconds; that is

$$T = f(N) = N + 40.$$

So, if a cricket chirps 20 times in 15 seconds, then the temperature must be given by

$$f(20) = 20 + 40 = 60°\text{F}.$$

What if it was known that the temperature was 70°F? How would we determine how many times a cricket would be expected to chirp in a 15-second time interval at this temperature? We could set $f(N) = 70$ and solve for n:

$$\begin{aligned} N + 40 &= 70 \\ N + 40 - 40 &= 70 - 40 \\ N &= 30. \end{aligned}$$

Thus at a temperature of 70°F, a cricket would be expected to chirp 30 times in a 15-second time interval.

In fact, if we wanted to know how many times a cricket would be expected to chirp in a 15-second time interval at any specific temperature T, we could set $f(N) = T$ and solve for N:

$$\begin{aligned} N + 40 &= T \\ N + 40 - 40 &= T - 40 \\ N &= T - 40. \end{aligned}$$

Now we have the number of times a cricket will chirp in a 15 second time interval written as a function of the air temperature:

$$N = g(T) = T - 40.$$

We have in fact found the **inverse function** of $f(n)$.

Inverse Functions

A function takes an input, does something to it, and produces a single output number. Given a specific output from a function, we may wonder what input resulted in this output. Informally, an *inverse function* can be thought of as a function's "undo button" (provided it exists). It takes the output the original function gave, does something to it, and the result is the input that we put in the original function to begin with.

The Inverse of a Function

An **inverse function** of a function $f(x)$ is a function $g(x)$ that satisfies both of the following conditions:

- $g(f(x)) = x$ for all x in the domain of $f(x)$
- $f(g(x)) = x$ for all x if the domain of $g(x)$

Each of the above bullet points describes the following behavior of an inverse: $f(x)$ takes the input x, does something to it, and produces an output. If we then put this output through the inverse funciton $g(x)$, it "undoes" the work done by the original function and takes us back to the initial input x.

Example 7.4.1. Show that $f(n) = N + 40$ and $g(T) = T - 40$ are inverse functions.

Solution. To show that $f(N)$ and $g(T)$ are inverse functions, we must verify that when the functions are composed in either order, the result is the input variable of the interior function in the composition. It can be seen that

$$\begin{aligned} f(g(T)) &= f(T - 40) \\ &= (T - 40) + 40 \\ &= T \end{aligned}$$

while

$$\begin{aligned} g(f(N)) &= g(N+40) \\ &= (N+40)-40 \\ &= N \end{aligned}$$

so $f(N)$ and $g(T)$ are indeed inverses.

Finding Inverse Functions Algebraically

If the ordered pair (x, y) is on the graph of the function f, then the ordered pair (y, x) is on the graph of its inverse. This gives us an idea of how to find an inverse function algebraically.

Finding the Inverse of a Function Algebraically

To find the inverse of a function $f(x)$:

1. Set $y = f(x)$.
2. Reverse the roles of x and y.
3. Solve the resulting equation for y.
4. Verify that your result from Step 3 is a function. If it is, then this is the inverse function $f^{-1}(x)$ of $f(x)$. If it is not, then $f(x)$ does not have an inverse function.

Example 7.4.2. If possible, find the inverse of each function. If not possible, state this to be the case.

(a) $f(x) = 5x - 7$

(b) $g(t) = 4\sqrt[3]{t-5} + 2$

(c) $h(x) = \dfrac{x+1}{x-2}$

(d) $w(z) = z^2$

Solution. (a) To find the inverse, we write $y = f(x)$, reverse the roles of x and y, and then solve for y:

$$\begin{aligned} y &= 5x - 7 && \longleftarrow \text{ replace } f(x) \text{ with } y \\ x &= 5y - 7 && \longleftarrow \text{ reverse roles of } x \text{ and } y \\ x + 7 &= 5y - 7 + 7 \\ x + 7 &= 5y \\ \frac{x+7}{5} &= \frac{\cancel{5}y}{\cancel{5}} \\ \frac{x+7}{5} &= y && \longleftarrow \text{ solve for } y \end{aligned}$$

The equation $y = \dfrac{x+7}{5}$ is a linear function, so $f(x)$ has an inverse function and it is given by $f^{-1}(x) = \dfrac{x+7}{5}$.

(b)

$$
\begin{aligned}
y &= 4\sqrt[3]{t-5}+2 \quad \longleftarrow \text{ replace } g(t) \text{ with } y\\
t &= 4\sqrt[3]{y-5}+2 \quad \longleftarrow \text{ reverse roles of } t \text{ and } y\\
t-2 &= 4\sqrt[3]{y-5}+2-2\\
t-2 &= 4\sqrt[3]{y-5}\\
\frac{t-2}{4} &= \frac{\cancel{4}\sqrt[3]{y-5}}{\cancel{4}}\\
\frac{t-2}{4} &= \sqrt[3]{y-5}\\
\left(\frac{t-2}{4}\right)^3 &= \left(\sqrt[3]{y-5}\right)^3\\
\left(\frac{t-2}{4}\right)^3 &= y-5\\
\left(\frac{t-2}{4}\right)^3+5 &= y-5+5\\
\left(\frac{t-2}{4}\right)^3+5 &= y \quad \longleftarrow \text{ solve for } y
\end{aligned}
$$

The equation $y = \left(\frac{t-2}{4}\right)^3 + 5$ is a transformation of the cubic function $y = t^3$ and so is itself a function. Hence $g(t)$ has an inverse function and it is given by $g^{-1}(t) = \left(\frac{t-2}{4}\right)^3 + 5$.

(c)

$$
\begin{aligned}
y &= \frac{x+1}{x-2} \quad \longleftarrow \text{ replace } h(x) \text{ with } y\\
x &= \frac{y+1}{y-2} \quad \longleftarrow \text{ reverse roles of } x \text{ and } y\\
x(y-2) &= \frac{y+1}{\cancel{y-2}}\cancel{(y-2)}\\
xy-2x &= y+1\\
xy-2x+2x &= y+1+2x\\
xy &= y+1+2x\\
xy-y &= y-y+1+2x\\
xy-y &= 1+2x\\
y(x-1) &= 1+2x\\
\frac{y\cancel{(x-1)}}{\cancel{x-1}} &= \frac{1+2x}{x-1}\\
y &= \frac{1+2x}{x-1} \quad \longleftarrow \text{ solve for } y
\end{aligned}
$$

Using a graphing calculator, it can be seen that $y = \dfrac{1+2x}{x-1}$ passes the vertical line test and is therefore a function. The inverse of $h(x)$ is $h^{-1}(x) = \dfrac{1+2x}{x-1}$.

(d)

$$\begin{aligned} y &= z^2 \quad \longleftarrow \text{ replace } w(z) \text{ with } y \\ z &= y^2 \quad \longleftarrow \text{ reverse roles of } z \text{ and } y \\ \pm\sqrt{z} &= \sqrt{y^2} \\ \pm\sqrt{z} &= y \quad \longleftarrow \text{ solve for } y \end{aligned}$$

The formula $y = \pm\sqrt{z}$ is *not* a function. Why? There are values of z which result in two different outputs for y; for instnace, $z = 1$ results in $y = \pm\sqrt{1} = \pm 1$. For the single input $z = 1$, there are two outputs, $y = -1$ and $y = 1$. For this reason, $w(z)$ does not have an inverse function.

Practice Problems for Section 7.4

In Problems 1-6, find $f(g(x))$ and $g(f(x))$ for the pair of functions $f(x)$ and $g(x)$ given. Use your results to determine whether $f(x)$ and $g(x)$ are inverses of each other.

1. $f(x) = x^2$ and $g(x) = \sqrt{x}$
2. $f(x) = 4x$ and $g(x) = \dfrac{4}{x}$
3. $f(x) = 2x + 5$ and $g(x) = \dfrac{x-5}{2}$
4. $f(x) = 3x - 1$ and $g(x) = \frac{1}{3}x + 1$
5. $f(x) = \sqrt[3]{x-2}$ and $g(x) = x^3 + 2$
6. $f(x) = \dfrac{3}{x-4}$ and $g(x) = \dfrac{3}{x} + 4$

In Problems 7-19, find the inverse of each function, if possible. If not possible, state this to be the case.

7. $f(x) = 5x$
8. $f(t) = \dfrac{2}{t}$
9. $f(x) = 3 - \frac{1}{2}x$
10. $f(w) = 5x^2$
11. $f(x) = 10 - 3x^3$
12. $f(x) = \sqrt[3]{x+1}$
13. $f(x) = 4(x+1)^2 - 7$
14. $f(x) = \dfrac{1}{3x-1}$
15. $f(x) = \dfrac{2x-5}{3x+1}$
16. $f(x) = 2^x$
17. $f(x) = 5e^x - 6$
18. $f(x) = \ln(x+1)$
19. $f(x) = \frac{1}{2}\log_6(x) - 10$

20. The function $f(x) = \frac{9}{5}x + 32$ is used to convert an input x in degrees Celsius to an output $f(x)$ in degrees Fahrenheit. Find the inverse function, which is the formula used to convert an input in degrees Fahrenheit to an output in degrees Celsius.

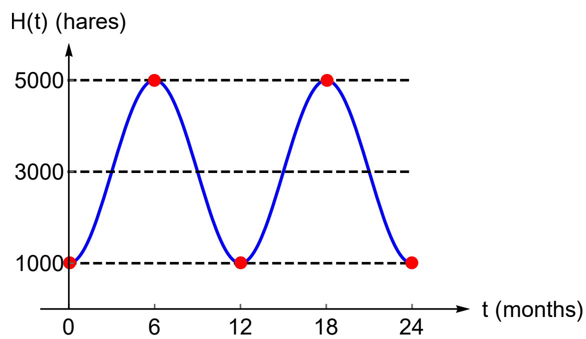
Chapter 8

Trigonometric Functions

8.1 Periodic Functions

Many functions encountered in mathematics and in real-life applications display oscillatory, wave-like, or *cyclical* behavior; that is, the same cycle of a constant duration keeps repeating over and over again. Examples of this include the volume of the air in your lungs as you breathe in and out, the temperature of a room that is regulated by a thermostat, the height of the water in a harbor between the tides, and so on. In mathematics such cyclical functions are called *periodic.*

Example 8.1.1. Each year a hare population, $H = H(t)$, in a national park changes with the seasons. The population is at its minimum of 1000 hares in January. In the summer, when the weather gets warm and the grass is green, the population grows in size to 5000. By the following January, the population decreases again to 1000 hares. The same cycle repeats every year. Let t be measured in months and $t = 0$ correspond to January. The graph of the population function $H(t)$ is as follows:



The behavior of this population function is cyclical — the function is periodic. The time needed for one full cycle to be executed is 12 months. We say the *period* of $H(t)$ is 12 months. Note that for every time t the size of the population 12 months later, at time $t + 12$, is exactly the same as the population at t.

A precise definition of a periodic function is as follows.

The Defintion of a Periodic Function

A nonconstant function f is called **periodic** if a positive number P exists such that

$$f(t) = f(t+P)$$

for all t in the domain of f. The smallest such P is called the **period** of f. If t is time, we can characterize the period as the shortest time needed for one full cycle of f to be executed.

The definition says that adding the period P to the input of the original function does not change the original output. In other words, the behavior of $f(t)$ on the interval $0 \leq t \leq P$ will be repeated on the interval $P \leq t \leq 2P$, and then again on the interval $2P \leq t \leq 3P$, and so on. The function executes the same cycle on every interval of duration P.

Besides the period, there are two other important numbers associated with a periodic function: the *amplitude*, the *equilibrium*, the second of which is closely related to the concept of the *midline*.

The Amplitude, Equilibrium, and Midline

Let $y = f(t)$ be a periodic function. Let y_{max} be the largest possible value of f and let y_{min} be the smallest possible value of f.

The **amplitude** is given by:

$$\text{amplitude} = \frac{y_{max} - y_{min}}{2}.$$

The midpoint between y_{max} and y_{min} is called the **equilibrium** of a periodic function and is given by:

$$\text{equilibrium} = \frac{y_{max} + y_{min}}{2}.$$

The **midline** is the horizontal line through the point $(0, \frac{y_{max}+y_{min}}{2})$ on the vertical axis; that is, the line with the equation:

$$y = \frac{y_{max} + y_{min}}{2}.$$

We say for short that the midline is the horizontal line "through the equilibrium."

In Example 8.1.1 the period is 12 months. The maximum value of H is $H_{max} = 5000$. The minimum value is $H_{min} = 1000$. Hence, the amplitude is

$$\frac{5000 - 1000}{2} = \frac{4000}{2} = 2000.$$

The equilibrium is

$$\frac{5000 + 1000}{2} = 3000.$$

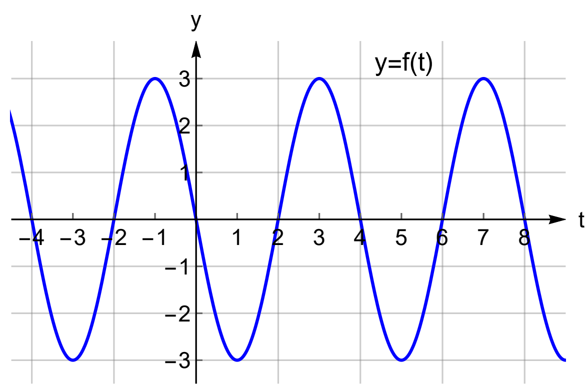
Hence, the midline is the horizontal line through 3000; that is, the line $H = 3000$. The graph of $H(t)$ above illustrates all three concepts.

We can characterize the amplitude, equilibrium and midline in intuitive terms as follows:

The Amplitude, Equilibrium, and Midline in Common Sense Terms

- If a periodic function attains its maximum once per cycle (that is, has one peak per cycle) then the **period** is the distance between consecutive peaks.
- The **equlibrium** is the value midway between the maximum and minimum values.
- The **midline** is the horizontal line through the equilibrium.
- The **amplitude** is the vertical distance from the midline to a peak.

Example 8.1.2. Find the period, amplitude, equilibrium and midline of the function $y = f(t)$ whose graph is given below.



Solution. We look carefully at the graph of the function. At $t = 0$ the function starts from the value $f(0) = 0$. As t increases, $f(t)$ decreases all the way to -3. Then $f(t)$ starts increasing, reaches 0 again at $t = 2$, continues increasing to the value 3, decreases back to 0 at $t = 4$. After $t = 4$, the same cycle repeats between $t = 4$ to $t = 8$: decreasing to -3, increasing to 0, increasing to 3, decreasing back to 0. The function executes the same cycle every 4 units of t. Hence, $f(t)$ is periodic with the period 4. To calculate the amplitude and the equilibrium, notice that

$$y_{min} = -3, \quad y_{max} = 3.$$

We then have:

$$\text{period} = 4,$$

$$\text{amplitude} = \frac{y_{max} - y_{min}}{2} = \frac{3 - (-3)}{2} = \frac{3 + 3}{2} = 3,$$

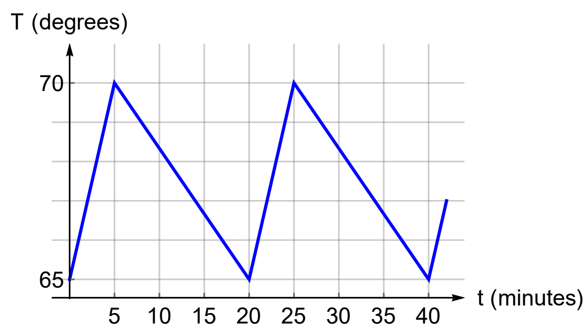
$$\text{equilibrium} = \frac{y_{max} + y_{min}}{2} = \frac{3 + (-3)}{2} = \frac{3 - 3}{2} = 0.$$

The midline is the horizontal line through the equilibrium. So it is the line given by the equation:

$$y = 0.$$

The function $f(t)$ oscillates around the equilibrium $y = 0$, varying between 3 units down and 3 units up from the equilibrium; that is, with amplitude 3. It executes one full cycle over any interval of duration 4.

Example 8.1.3. The temperature of a room, $T(t)$, in $^\circ F$, is regulated by a thermostat. The thermostat triggers the heat to go on when the temperature drops to $65^\circ F$ and shuts the heat at $70^\circ F$. Below is the graph of the temperature function $T(t)$, t is measured in minutes.



(a) Find the period, amplitude, equilibrium, and the midline.

(b) Suppose that $t = 0$ corresponds to noon 12 p.m.. What is the temperature in the room at 12:45 p.m.?

(c) Sketch the midline on the graph of the function.

Solution. (a) The function $T(t)$ reaches its peak once a cycle. The distance between two peaks is 20 minutes. Hence, the period is 20 minutes. The minimum and maximum values of the temperature are $T_{max} = 70$ and $T_{min} = 65$ respectively. We calculate:

$$\text{period} = 20,$$

$$\text{amplitude} = \frac{T_{max} - T_{min}}{2} = \frac{70 - 65}{2} = \frac{5}{2} = 2.5,$$

$$\text{equilibrium} = \frac{T_{max} + T_{min}}{2} = \frac{70 + 65}{2} = \frac{135}{2} = 67.5.$$

The midline is the horizontal line through the equilibrium; that is, the line with the equation:

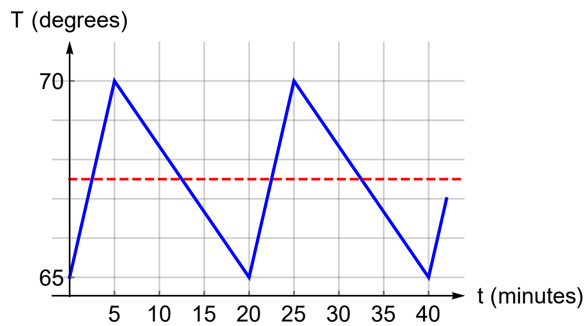
$$T = 67.5.$$

(b) For every t, $T(t+20) = T(t)$ as 20 is the period of the function. We see from the graph that $T(25) = 70°F$. Hence, $T(45) = T(25+20) = T(25) = 70°F$. Of course, 12:45 P.M. is exactly 20 minutes after 12:25 p.m.. So the temperature at 12:45 P.M. is the same as at 12:25 P.M. which is $70°F$.

(c) The midline is the horizontal line:

$$T = 67.5.$$

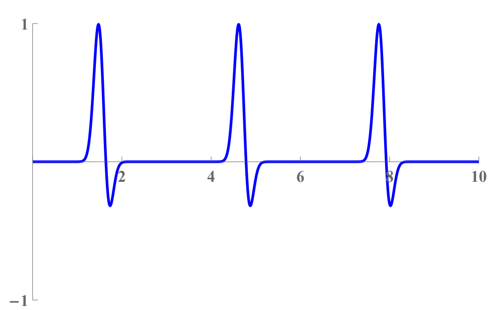
Here is the midline in red:



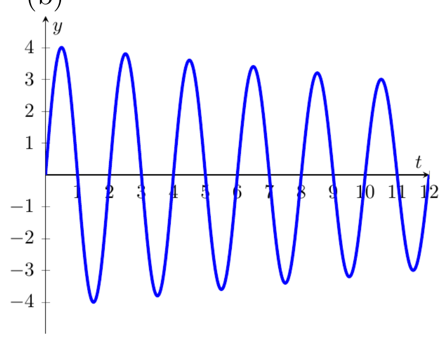
Practice Problems for Section 8.1

1. Determine whether each function graphed below is a periodic function. For those that are periodic, approximate the amplitude, midline, and period.

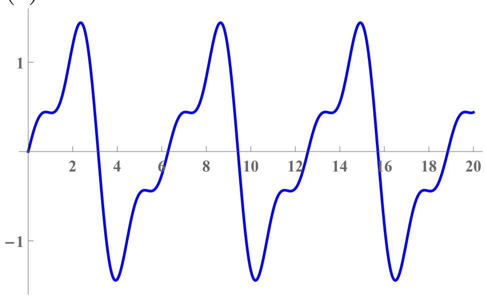
(a)



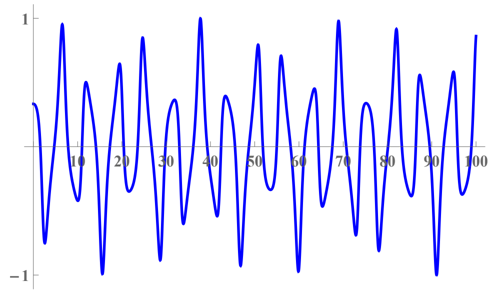
(b)



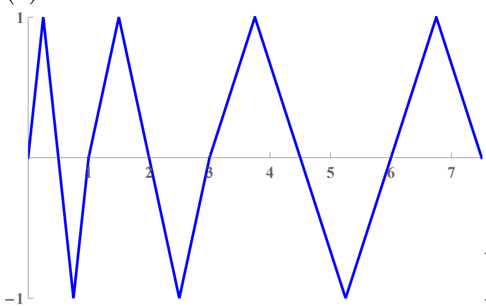
(c)



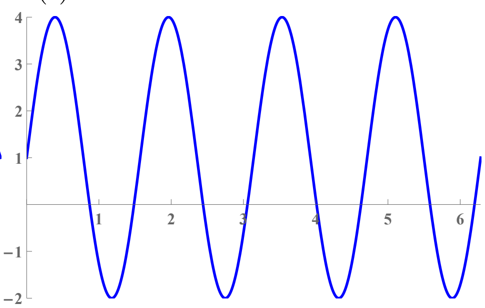
(d)



(e)



(f)



2. Each day, the tide in a harbor continuously goes in and out, raising and lowering a boat anchored there. At low tide, the boat is only 2 meters above the ocean floor. Six hours later, at peak high tide, the boat is 20 meters above the ocean floor. Six hours after peak high tide, it is low tide again. Suppose the boat is at high tide at midnight.

   (a) Sketch and label a periodic function $D(t)$ modeling the boat's distance above the ocean floor as a function of time t hours since midnight.

   (b) Identify the period, amplitude, and midline of the periodic function.

3. You decide to ride the ferris wheel at the local carnival. You are 3 feet above the ground at the bottom of the ferris wheel and 28 feet above the ground at the top. It takes 8 seconds for the you to reach the maximum height from the minimum height and 8 seconds to reach the minimum height from the maximum height. Suppose you are at the bottom of the ride at time $t = 0$ seconds.

   (a) Sketch and label a periodic function $H(t)$ modeling your height above the ground t seconds into your ferris wheel ride.

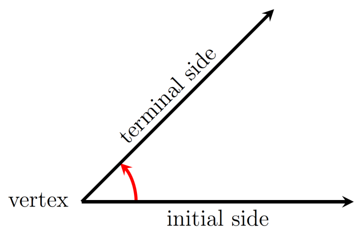
   (b) Identify the period, amplitude, and midline of the periodic function.

8.2 Angles Using the Unit Circle, Radian Measure

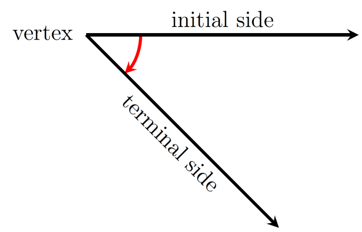
We are working toward defining the two important trigonometric functions $f(t) = \sin(t)$ and $f(t) = \cos(t)$. These functions, referred to as the sine function and the cosine function respectively, are periodic functions which are commonly used in applied sciences to model periodic processes.

The first step is to define the trigonometric ratios $\sin(\alpha)$ and $\cos(\alpha)$ for any angle α, positive or negative, large or small. We also have to learn how to measure angles in radians rather than in degrees. In this section we define positive and negative angles of any size and radian measure.

For a given angle, we designate one side as the initial side and the other as the terminal side. We imagine the angle being swept out starting from the initial side and ending at the terminal side, as shown below. When the rotation from the initial side to terminal side is counterclockwise, we say that the angle is positive; when the rotation is clockwise, we say that the angle is negative.

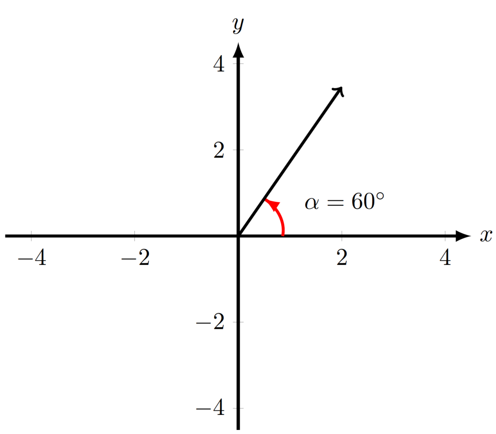


$45°$ angle
positive as rotation is counterclockwise

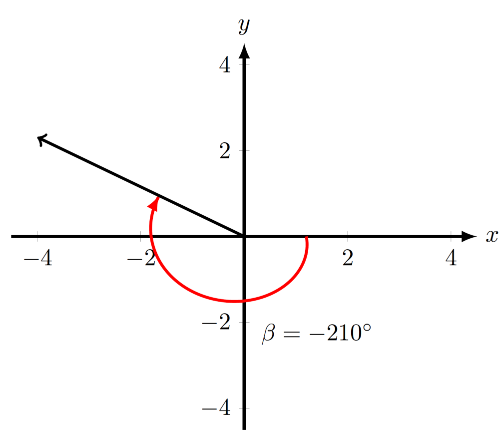


$-45°$ angle
negative as rotation is clockwise

We say that an angle is in *standard position* if it is located on the xy-plane with the vertex at the origin $(0, 0)$ and the initial side aligned with the positive x-axis. Again, counterclockwise rotations generate positive angles while clockwise rotations yield negative angles:

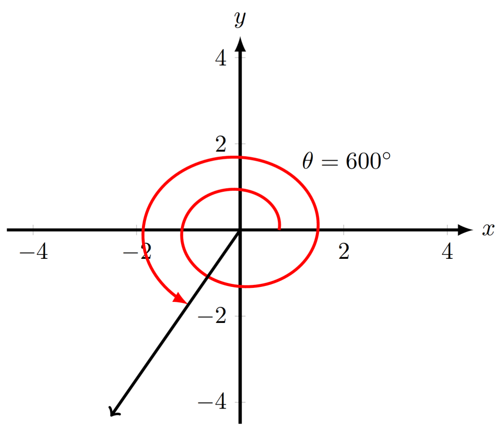


angle $\alpha = 60°$ in standard position

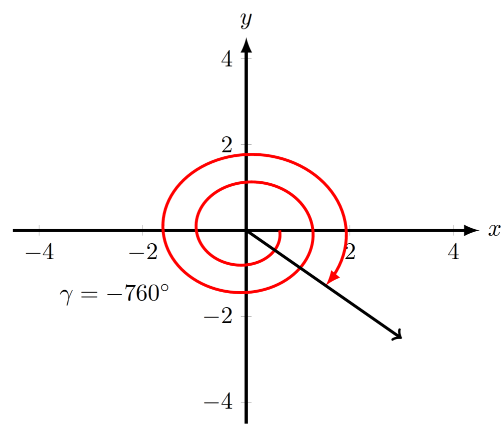


angle $\beta = -210°$ in standard position

We can rotate around the origin as many times as we want and obtain angles greater than $360°$ and smaller than $-360°$:



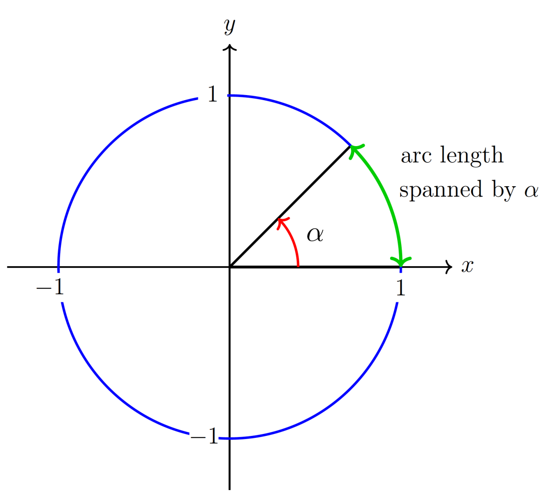
angle $\theta = 600°$ in standard position



angle $\gamma = -760°$ in standard position

Definition of Radian Measure

The radian measure of a positive angle α is equal to the length of the arc spanned by the angle on the circle of radius 1 centered at the vertex of the angle:



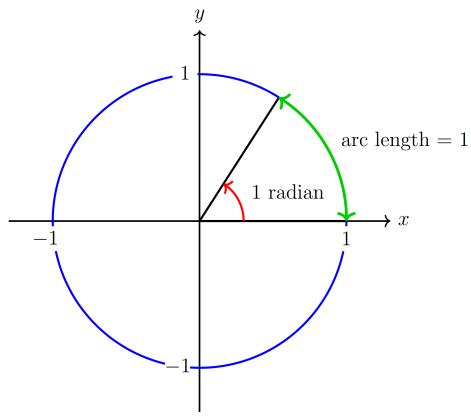
The radian measure of a negative angle is equal to minus the length of the arc spanned by the angle.

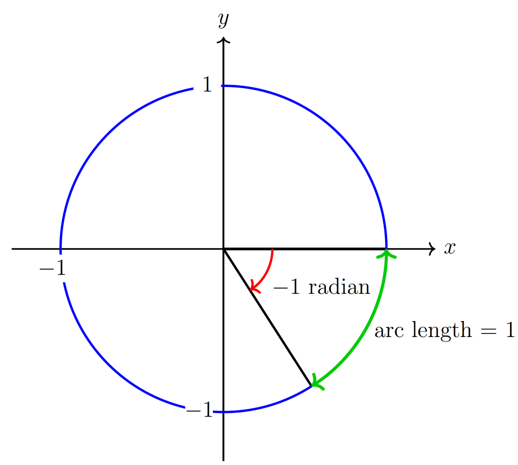
Recall that the *unit circle* on the xy-plane is the circle of radius 1 centered at the origin $(0, 0)$. Here is the precise definition of one radian.

Defintion of Radian Measure

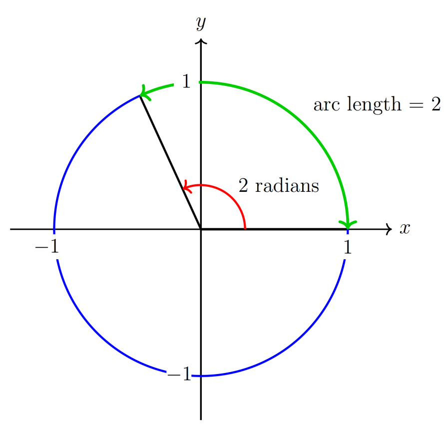
- An angle of **1 radian** is the angle in standard position, in the counterclockwise direction, which spans the arc of length 1 on the unit circle.
- An angle of -1 **radian** is the angle in standard position, in the clockwise direction, which spans the arc of length 1 on the unit circle.

The radius and the arc must be measured in the same units of length.

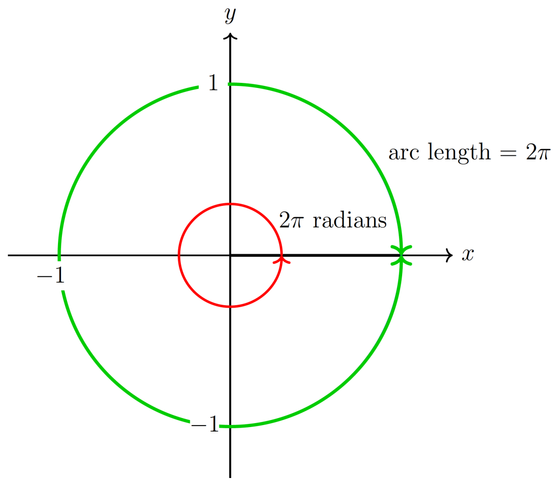




The angle of 1 radian spans the arc of length 1 on the unit circle — the length of the arc and the length of the radius are the same. Imagine "picking up" the radius of the unit circle like a yardstick and wrapping it over a piece of the circle to obtain the arc of the same length as the radius. If you measure the arc of length equal to twice the radius, the arc will correspond to the angle of 2 radians and so on:



The first thing we notice is that the angle of 1 radian, denoted by 1 rad, is a relatively large angle while the angle of $1°$ is a very small angle. To understand why it is so, let's look at the full angle of $360°$ in terms of radians. The circumference of a circle of radius r is $C = 2\pi r$. The circumference — the arc length — of the unit circle is $C = 2\pi$. Therefore, the angle that spans the whole circle, one complete counterclockwise revolution, is the angle of 2π radians. That angle in degrees is, of course, $360°$:



The full angle of $360°$ is $2\pi \approx 6.28$ radians. Hence, we have the following relationship between degrees and radians:

$$360° = 2\pi \text{ radians.}$$

The latter equality gives the following **conversion formulas**:

$$1 \text{ degree} = \tfrac{\pi}{180} \text{ radians}, \quad 1 \text{ radian} = \tfrac{180}{\pi} \text{ degrees.}$$

Note that $1° \approx 0.0175$ rad and 1 rad $\approx 57.296°$. 1 radian is equal to approximately $57°$.

Example 8.2.1. Convert the following angles from degrees to radians.

(a) $45°$ (b) $90°$ (c) $180°$ (d) $17°$

Solution. We use conversion formulas:

(a) $45° = 45 \cdot \frac{\pi}{180} = \frac{\pi}{4} \approx 0.785$ rad.

(b) $90° = 90 \cdot \frac{\pi}{180} = \frac{\pi}{2} \approx 1.57$ rad.

(c) $180° = 180 \cdot \frac{\pi}{180} = \pi \approx 3.142$ rad.

(d) $17° = 17 \cdot \frac{\pi}{180} \approx 0.297$ rad.

Example 8.2.2. Convert the following angles from radians to degrees.

(a) $-\frac{\pi}{3}$ rad (b) $\frac{\pi}{6}$ rad (c) $\frac{3\pi}{2}$ rad (d) -2.5 rad

Solution. We use conversion formulas:

(a) $-\frac{\pi}{3}$ rad $= -\frac{\pi}{3} \cdot \frac{180}{\pi} = -60°$.

(b) $\frac{\pi}{6}$ rad$=\frac{\pi}{6} \cdot \frac{180}{\pi} = 30°$.

(c) $\frac{3\pi}{2}$ rad $=\frac{3\pi}{2} \cdot \frac{180}{\pi} = 270°$.

(d) -2.5 rad $=-2.5 \cdot \frac{180}{\pi} \approx -143.24°$.

Here is a list of some frequently used angles and their measure in degrees and in radians.

Degrees	**Radians**	**Radians Approx.**
30°	$\frac{\pi}{6}$	0.52
45°	$\frac{\pi}{4}$	0.79
60°	$\frac{\pi}{3}$	1.05
90°	$\frac{\pi}{2}$	1.57
135°	$\frac{3\pi}{4}$	2.36
180°	π	3.14
225°	$\frac{5\pi}{4}$	3.93
270°	$\frac{3\pi}{2}$	4.71
315°	$\frac{7\pi}{4}$	5.50
360°	2π	6.28

The figure below shows a few angles on the unit circle together with their radian measure. From now on, you should try to get used to radian measure and think about angles in terms of radians rather than degrees.

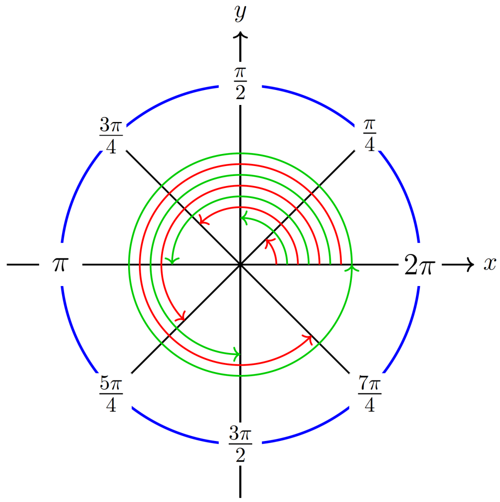


Figure 8.1

Example 8.2.3. In which quadrant is the angle of 2 radians? An angle of 5 radians? (In other words, in which quadrants are the terminal sides of these angles?)

Solution. Refer to Figure 8.1. The second quadrant includes angles between $\pi/2$ and π; that is, between $\pi/2 \approx 1.57$ and $\pi \approx 3.14$ radians. So the angle of 2 radians lies in the second quadrant. The angle of 5 radians is between $3\pi/2 \approx 4.71$ and $2\pi \approx 6.28$ radians. Thus, the angle of 5 radians is in the fourth quadrant.

An angle of p radians spans an arc of length p on the unit circle. By similarity of circular sectors, the same angle of p radians spans the arc of length $p \cdot r$ on the circle of radius r:

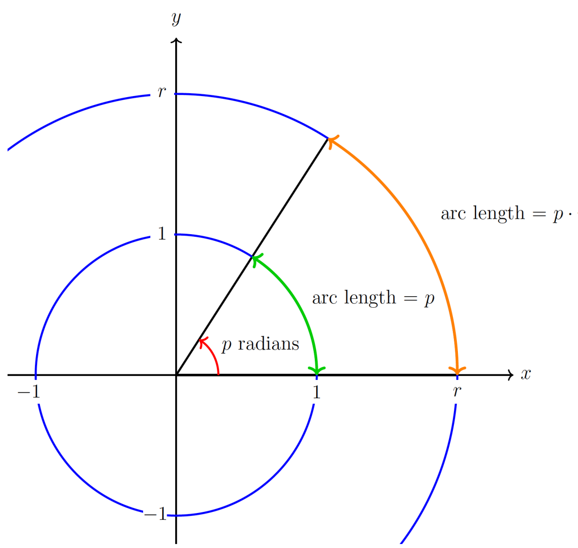


Figure 8.2

The length, s, of the arc spanned in a circle of radius r by an angle of p radians is:

$$s = p \cdot r.$$

Example 8.2.4. Find the arc length s spanned by an angle of $\pi/2$ radians on the circle of radius 5 centimeters.

Solution. According to the formula above, $s = (\pi/2) \cdot 5 \approx 7.854$ centimeters. (Remember, units of radius and arc length are the same.)

Practice Problems for Section 8.2

For Problems 1-4, convert the following angles from degrees to radians. Give both exact answers and, when appropriate, approximations to three decimal places. Also identify which quadrant each angle lies in.

1. $27°$
2. $-10°$
3. $600°$
4. $-562°$

For Problems 5-8, convert the following angles from radians to degrees. Give both exact answers and, when appropriate, approximations to three decimal places. Also identify which quadrant each angle lies in.

5. $\frac{\pi}{18}$ radians
6. $-\frac{\pi}{10}$ radians
7. 8 radians
8. $\frac{6\pi}{5}$ radians

9. What is the arc length spanned by the angle $\frac{3\pi}{4}$ radians on the circle of radius 4 inches?

10. What is the arc length spanned by the angle $300°$ on the circle of radius 2 centimeters?

8.3 Values of Sine and Cosine for Angles on the Unit Circle

For every angle α we define the values $\sin(\alpha)$ and $\cos(\alpha)$ using the coordinates of the endpoint of the terminal side when α is an angle in standard position on the unit circle. Take an angle α in standard position. Let $P = (x, y)$ be the point where the terminal side of the angle intersects the unit circle:

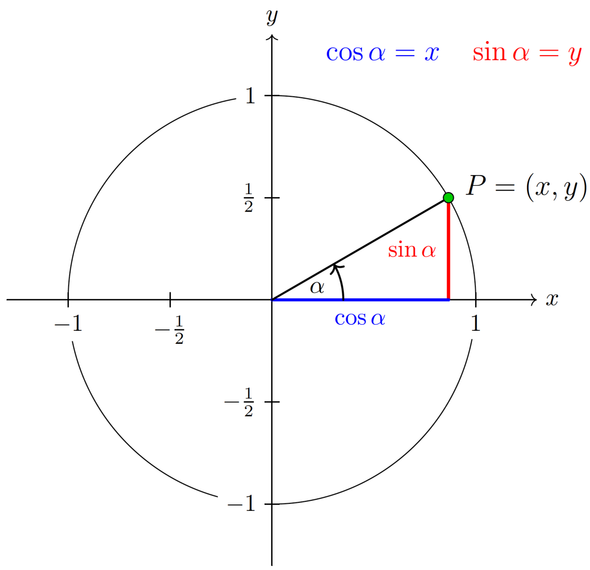


Figure 8.3

Trigonometric Ratios $\sin(\alpha)$ **and** $\cos(\alpha)$

Let $-\infty < \alpha < +\infty$ be an angle in standard position on the xy-plane. Let $P = (x, y)$ be the point of intersection of its terminal side with the unit circle. We define the values $\sin(\alpha)$ and $\cos(\alpha)$ as follows:

$$\cos(\alpha) = x, \quad \sin(\alpha) = y.$$

The following identity holds for all angles α:

Pythagorean Identity

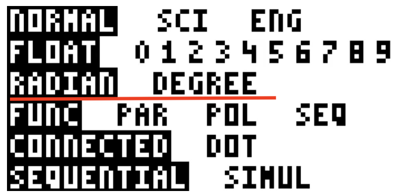
For every $-\infty < \alpha < +\infty$:

$$\sin^2(\alpha) + \cos^2(\alpha) = 1.$$

At this point, the angle α can be measured in radians or in degrees but we must always be aware and careful which measure we are using: $\sin(1)$ when 1 is in radians and $\sin(1)$ when 1 is in degrees have completely different values. Indeed, according to the conversion formulas from the previous sections, 1 radian is approximately 57.3 degrees so 1 radian and 1 degree are not even close.

Example 8.3.1. Use your calculator to compare values of $\sin(1)$ where 1 is 1 radian and $\sin(1)$ where 1 is 1 degree.

Solution. Your calculator can give you values of $\sin(1)$ for 1 degree and 1 radian provided you set it in the right **MODE**. Here is a screen shot from TI-84 calculator:



Set your calculator in RADIANS. You obtain:

$$\sin(1) = \sin(1 \text{ radian}) \approx 0.841.$$

Set your calculator in DEGREES. You obtain:

$$\sin(1) = \sin(1 \text{ degree}) \approx 0.017.$$

As you can see the values are very different.

Example 8.3.2. Use the definition above to find values of $\sin(\theta)$, $\cos(\theta)$ for $\theta = 0$, $\pi/2$, π, $3\pi/2$, 2π, where θ is in radians.

Solution. We look at Figure 8.1 where the position of many angles in radians on the unit circle is shown.

For each value of θ we find the corresponding point $P = (x, y)$ and determine values of the coordinates x and y. We know that $\cos(\theta) = x$, $\sin(\theta) = y$.

The angle 0 radians specifies the point $P = (1, 0)$. Hence, $\sin(0) = 0$, $\cos(0) = 1$.

The angle $\frac{\pi}{2}$ (equal to $90°$) gives $P = (0, 1)$. Hence, $\sin(\frac{\pi}{2}) = 1$, $\cos(\frac{\pi}{2}) = 0$. (See Figure 8.4.)

For $\theta = \pi$ the point $P = (-1, 0)$. So $\sin(\pi) = 0$, $\cos(\pi) = -1$.

For $\frac{3\pi}{2}$ the point $P = (0, -1)$. So $\sin(\pi) = -1$, $\cos(\pi) = 0$.

Observe that at $\theta = 2\pi$ we are back at point $P = (1, 0)$: the angles $\theta = 0$ and $\theta = 2\pi$ have the same terminal side. Hence, $\sin(2\pi) = 0$, $\cos(2\pi) = 1$. As we increase θ beyond 2π the cycle of values for $\sin(\theta)$ and $\cos(\theta)$ repeats on the interval $[2\pi, 4\pi]$ and it keeps repeating on each interval of the length 2π.

The following is a helpful picture for $\theta = \frac{\pi}{2}$ and $\theta = \pi$:

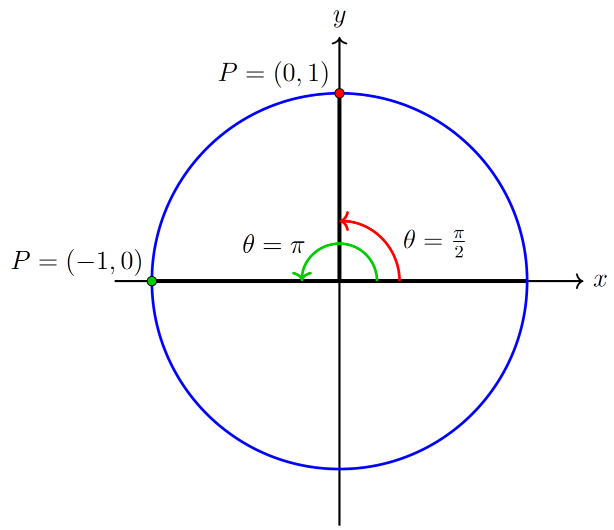


Figure 8.4

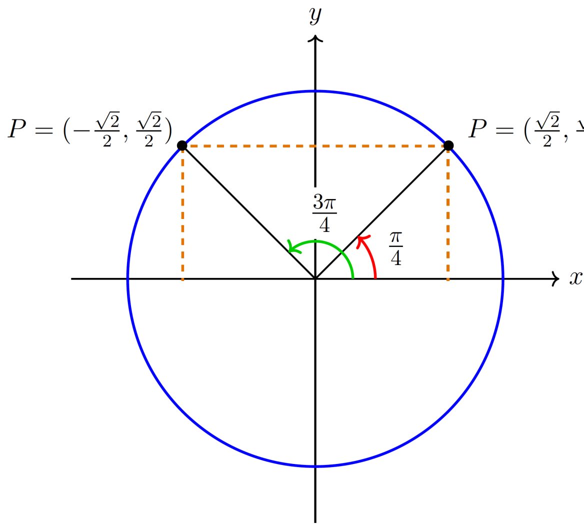
If you studied trigonometric ratios in the right triangle, you know the exact values of sine and cosine for a few special angles. Here they are, including the values of the angles in radians:

α(degrees)	α(radians)	$\cos(\alpha)$	$\sin(\alpha)$
$0°$	0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$60°$	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$90°$	$\frac{\pi}{2}$	0	1

We can use the values in the table and the positions of angles on the unit circle, to find sine and cosine of many other angles.

Example 8.3.3. Find $\sin\left(\frac{3\pi}{4}\right)$ and $\cos\left(\frac{3\pi}{4}\right)$.

Solution. We have to locate the terminal side of the angle $\frac{3\pi}{4} = 135° = 90° + 45°$ and find the point of intersection of the terminal side with the unit circle. The terminal side of the angle is located in the second quadrant and it is symmetric over the y-axis to the terminal side of the angle $\frac{\pi}{4} = 45°$:



The point of intersection of the terminal side of $\frac{\pi}{4}$ with the unit circle is $P = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as $x = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $y = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. (See the table.) By symmetry, the point of intersection of the terminal side of $\frac{3\pi}{4}$ with the unit circle is $P = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Hence: $\cos\left(\frac{3\pi}{4}\right) = x = -\frac{\sqrt{2}}{2}$ and $\sin\left(\frac{3\pi}{4}\right) = y = \frac{\sqrt{2}}{2}$.

Practice Problems for Section 8.3

In Problems 1-10, find the exact value without using a calculator.

1. $\sin\left(\frac{5\pi}{4}\right)$
2. $\cos\left(\frac{5\pi}{4}\right)$
3. $\cos\left(\frac{2\pi}{3}\right)$
4. $\sin\left(\frac{2\pi}{3}\right)$
5. $\sin(-90°)$
6. $\cos(-2\pi)$
7. $\cos\left(-\frac{\pi}{6}\right)$
8. $\sin\left(-\frac{\pi}{6}\right)$
9. $\sin(240°)$
10. $\cos(240°)$

8.4 Sine and Cosine Functions

Finally we are in a position to define the two important trigonometric functions $f(t) = \sin(t)$ and $g(t) = \cos(t)$. We will use the trigonometric ratios $\sin(\alpha)$ and $\cos(\alpha)$ to define values of the functions at any given t although in applications the functions are often detached from their unit circle meaning. Here is the definition.

Sine and Cosine Functions

Let t be a real number. Take the angle of t **radians** in standard position. The values of the sine and cosine functions at t are defined as:

$$f(t) = \sin(t), \quad g(t) = \cos(t)$$

where $\sin(t)$ and $\cos(t)$ are the trigonometric ratios of the angle t as described in the last section.

Graphs of Sine and Cosine Functions

What is the graph of the sine function $f(t) = \sin(t)$? We know that for the angle of t radians in standard position, $\sin(t)$ is the y coordinate of the point $P = (x, y)$ at which the terminal side intersects the unit circle (See Figure 8.3). Let's start from $t = 0$ and examine how the y coordinate of the point of intersection changes as we increase t. Look at Figure 8.5. Points of intersection of the terminal sides with the unit circle for the angles depicted on the figure are marked by red dots.

At $t = 0$ the terminal side of the angle 0 radians coincides with the initial side; the point of intersection is $P = (x, y) = (1, 0)$. Hence, $y = 0$ and $\sin(0) = 0$. Let's increase t counterclockwise as shown on Figure 8.5. For an angle whose terminal side is in the first quadrant, the y coordinate is between 0 and 1 and it increases as t increases. When the angle t reaches $\frac{\pi}{2} = 90°$, the terminal side is vertical, the point of intersection with the unit circle is $P = (0, 1)$; the y coordinate is $y = 1$. Hence, $\sin(\frac{\pi}{2}) = 1$. Look at the graph in Figure 8.6. For t between 0 and $\frac{\pi}{2}$ the function $\sin(t)$ increases from 0 to 1. Past $t = \frac{\pi}{2}$, $\sin(t)$ decreases when t increases until it hits 0 at $t = \pi$.

Continuing to reference Figure 8.5, we see that when we keep increasing t past $\frac{\pi}{2}$, the terminal side is in the second quadrant and the y coordinate is now decreasing until it is 0 at $t = \pi = 180°$. Let's keep increasing t past π. The terminal side is now in the third quadrant. The y coordinate is now negative and is getting "more and more negative" until it hits -1 at $\frac{3\pi}{2}$. Note how the graph of $\sin(t)$ in Figure 8.6 reflects this behavior. When the terminal side of the angle t is in the forth quadrant the y coordinate is getting "less and less negative" when t increases until at $t = 2\pi = 360°$ the terminal side is aligned again with the positive x-axis with $y = 0$. If we keep increasing t past 2π the same cycle of changes in y repeats on the interval $2\pi < t < 4\pi$. Therefore, the values of $\sin(t)$ repeat for $2\pi < t < 4\pi$. And so on.

To obtain the graph of the function $\cos(t)$, we follow changes in the x-coordinate of the intersection point $P = (x, y)$ as the angle t changes. We easily obtain the graph of $g(t) = \cos(t)$ given in Figure 8.7.

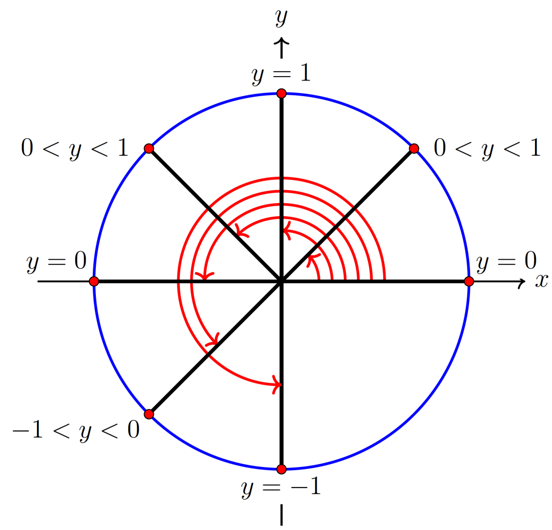


Figure 8.5

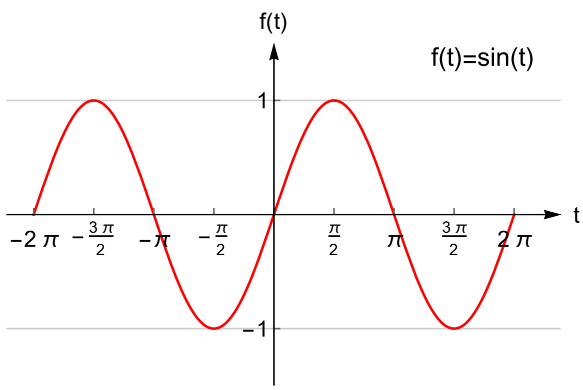


Figure 8.6

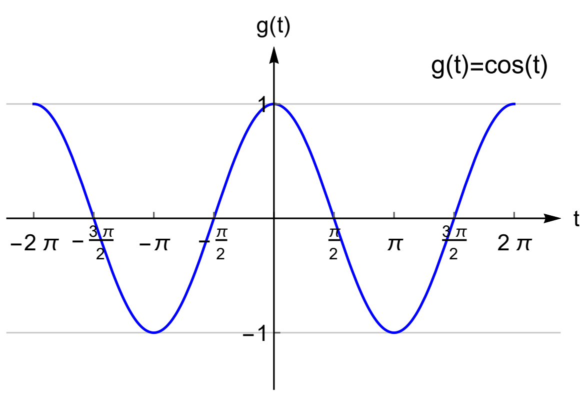


Figure 8.7

As we see from the definition and from the graphs of the sine and cosine functions, both functions $\sin(t)$ and $\cos(t)$ are periodic with period 2π:

$$\sin(t + 2\pi) = \sin(t), \quad \cos(t + 2\pi) = \cos(t).$$

The amplitude of both functions is 1, the equilibrium is 0, and the midline is the x-axis. This can be verified by observing that

$$f_{max} = 1, \quad f_{min} = -1, \quad g_{max} = 1, \quad g_{min} = -1.$$

Thus the amplitude and equilibrium for sine and cosine are:

$$\text{amplitude} = \frac{1 - (-1)}{2} = 1, \quad \text{equilibrium} = \frac{1 + (-1)}{2} = 0.$$

Before we leave the unit circle and the definition of the sine and cosine functions, let's make a simple observation about properties of the sine and cosine functions:

$$\sin(-t) = -\sin(t), \quad \cos(-t) = \cos(t).$$

Indeed, if angle t changes sign, the y coordinate of the intersection point of its terminal side with the unit circle changes sign while the x coordinate stays the same.

You can use your graphing calculator to graph the sine and cosine functions but you have to remember to set your calculator in radians and not in degrees. The values $\sin(t)$ and $\cos(t)$ were defined by interpreting t as an angle in radians.

Transforming Sine and Cosine Functions — Basic Ideas

It took a lot of work to define the sine and cosine functions. The good news is that once we have them, they are easy to use. All that we usually need to remember are the graphs of the functions $y = \sin(t)$ and $y = \cos(t)$ as given in Figures 8.6 and 8.7.

Speaking of graphs, notice that the function $\cos(t)$ starts from its maximum value at $t = 0$, $\cos(0) = 1$. The sine function on the other hand starts at $t = 0$ from its equilibrium value, $\sin(0) = 0$. These observations will be important in the next section when we try to match given periodic functions with a transformed sine or cosine function.

We used the unit circle and interpreted t as an angle to **define** the sine and cosine functions. In applications, we will typically get away from this interpretation. The independent variable t will most often be time and we will use the sine and cosine functions as convenient periodic functions that are useful for modeling periodic processes. We will use the notation $y = \sin(t)$ and $y = \cos(t)$ where y denotes the dependent variable and is no longer related to coordinates on the unit circle.

The sine and cosine functions, $y = \sin(t)$ and $y = \cos(t)$, wouldn't be very useful for modeling real-life periodic processes if we couldn't transform them to change the period, amplitude, or midline. How many real-life periodic processes have the period of exactly 2π, amplitude 1, and equilibrium 0? Not many. Fortunately, we can easily change all three numbers by vertical scaling and shifting and horizontal scaling.

Let's begin with some examples.

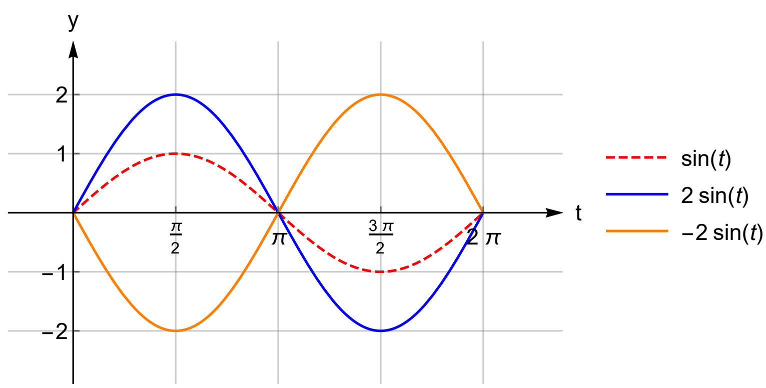
Changing the Amplitude

The amplitude of the functions $y = \sin(t)$ and $y = \cos(t)$ can be changed by vertical scaling.

Example 8.4.1. Find the period, amplitude, and midline of each of the functions. Compare their graphs to the graph of $y = \sin(t)$.

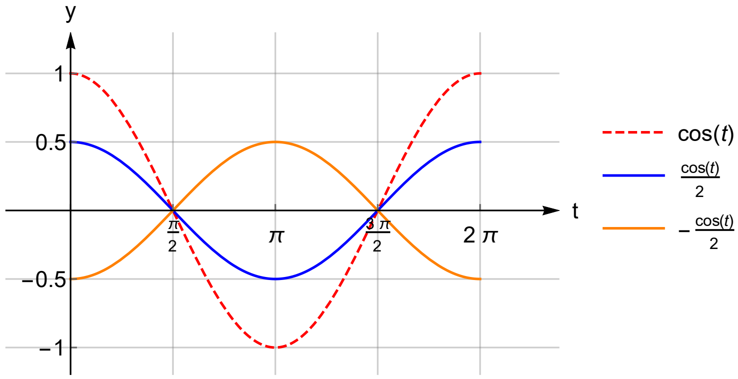
(a) $y = 2\sin(t)$ (b) $y = -2\sin(t)$

Solution. We learned in the previous chapter that multiplying the output by a constant corresponds to vertical scaling of the graph. Here are the graphs of the functions in (a), (b) and the original function $y = \sin(t)$ in one coordinate system:



The amplitude of both $y = 2\sin(t)$ and $y = -2\sin(t)$ is 2. The period is 2π and the midline is $y = 0$. Vertical scaling changes the amplitude only. If we multiply by a negative constant, we additionally have the reflection of the scaled graph over the t-axis.

We can change the amplitude of $y = \cos(t)$ by vertical scaling as well. Here is an example:



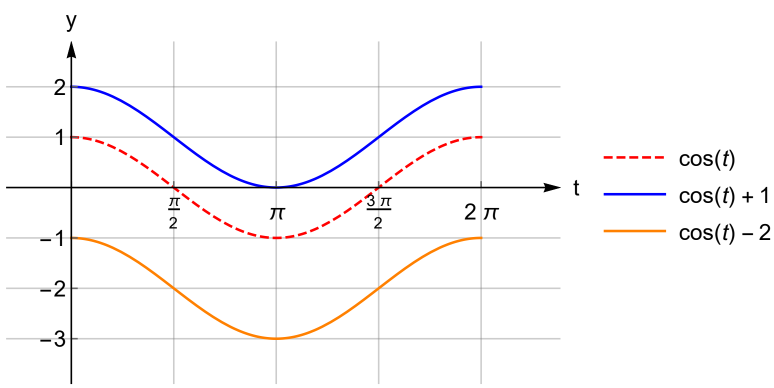
Changing the Midline

Predictably, vertical shifting doesn't change the amplitude or the period. It changes the equilibrium and the midline.

Example 8.4.2. Find the midline for

(a) $y = \cos(t) + 1$ (b) $y = \cos(t) - 2$

Solution. For (a) the midline is $y = 1$. For (b), the midline is $y = -2$. Vertical shift, shifts the midline accordingly. Here are graphs of the functions in (a) and (b) as well as the cosine itself:



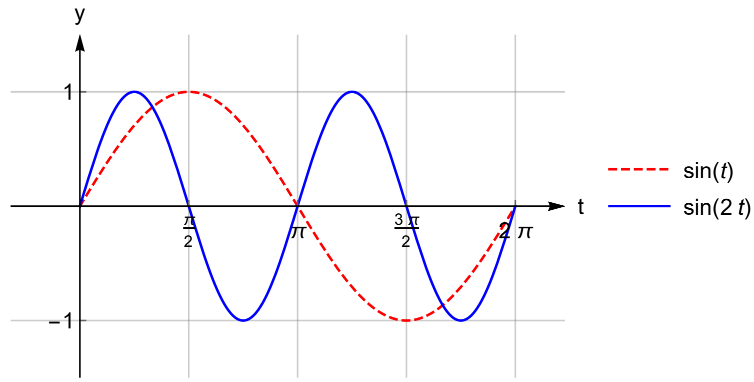
Changing the Period

How do we change the period of the sine and cosine function? We have to apply horizontal scaling.

Example 8.4.3. Find the period of the following functions:

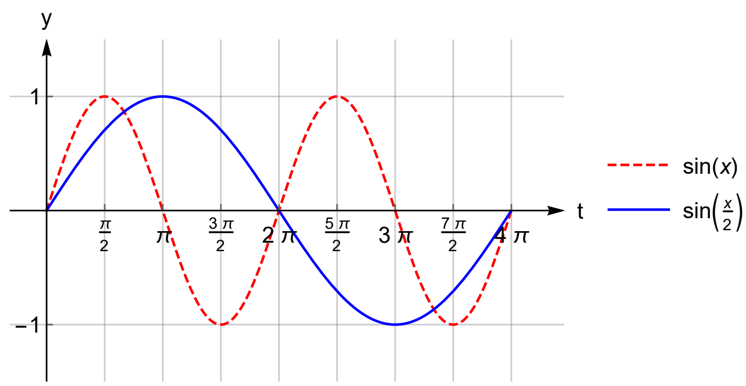
(a) $y = \sin(2t)$ (b) $y = \sin\left(\frac{1}{2}x\right)$.

Solution. (a) Lets look at the graphs of $y = \sin(2t)$ and $y = \sin(t)$:



We see from the graph that the function $y = \sin(2t)$ executes the full cycle — from 0 to 1, to 0, to -1, back to 0 — between $t = 0$ and $t = \pi$. Hence, the period of $y = \sin(2t)$ is π. The amplitude of the function is still 1 and the midline $y = 0$.

(b) The graph of $y = \sin\left(\frac{1}{2}x\right)$ clearly shows that the function executes one full cycle between $x = 0$ and $x = 4\pi$:

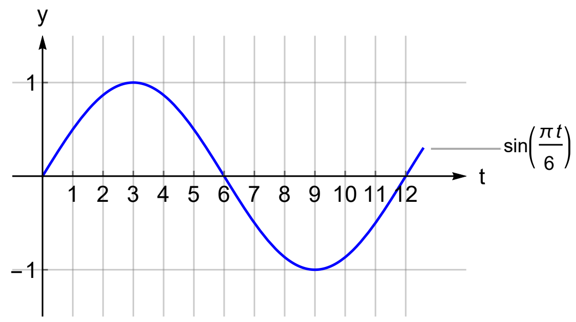


The period of $y = \sin(\frac{1}{2}x)$ is 4π.

Suppose that we want a periodic function with a period 12. (The first periodic function considered in this chapter had period 12.) Can we find such a function? Yes. Soon we will learn a systematic technique of doing it. For now let's just have an example:

Example 8.4.4. Consider the function $y = \sin\left(\frac{\pi}{6}t\right)$. What is the period of the function?

Let's look at the graph of the function:



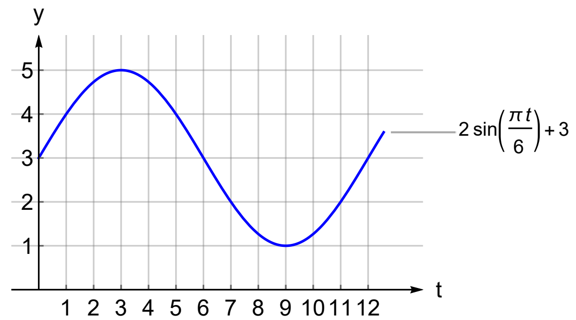
It appears that the function executes one full cycle between $t = 0$ and $t = 12$. The period of the function is 12.

We are getting a good idea of how to manipulate the sine and cosine functions to change the period, amplitude, and midline. We can always combine transformations to change all three quantities.

Example 8.4.5. Consider the function:

$$p(t) = 2\sin\left(\frac{\pi}{6}t\right) + 3$$

The graph of the function is:



What is the period, amplitude and midline of the function $p(t)$? What transformations have been applied to the sine function $\sin(t)$ to obtain $p(t)$?

Solution. Clearly we are looking at one cycle of a transformed sine function. So the period is 12. Since $p_{max} = 5$, $p_{min} = 1$, the amplitude is 2 and the midline is $y = 3$.

We applied horizontal scaling by the factor of $\frac{\pi}{6}$, vertical scaling by the factor of 2, and a vertical shift by 3.

Practice Problems for Section 8.4

For Problems 1-10, use a graph to identify the period, amplitude, and midline of the given function. Also identify what transformations have been applied to either $f(t) = \cos(t)$ or $f(t) = \sin(t)$ to obtain the given function.

1. $y = 5\cos(t)$
2. $y = -4\sin(t)$
3. $y = \sin(5t)$
4. $y = \cos\left(\frac{\pi}{3}t\right)$
5. $y = \cos(t) + 10$
6. $y = \sin(t) - 6$
7. $y = \sin\left(\frac{3}{5}t\right)$
8. $y = 2\cos(t) - 6$
9. $y = -3\sin(2t)$
10. $y = 10\sin\left(\frac{\pi}{2}t\right) + 4$

8.5 Families of Sine and Cosine Functions, Modeling

The examples of transforming the sine and cosine functions in the last section are illuminating but they do not give us a systematic and clear way of matching a transformed sine or cosine formula with a given periodic function—a periodic function that might have arisen in real-life applications.

Here are specific rules for transforming the sine and cosine functions through vertical scaling and shifting and horizontal scaling. Note that we will not consider horizontal shifting in this section.

Transforming Sine and Cosine Functions

Let constants A, B, C be given, $B > 0$. The functions

$$y = A\sin(Bt) + C \ , \ y = A\cos(Bt) + C$$

are periodic with

$$\text{amplitude} = |A| \quad , \quad \text{period} = \frac{2\pi}{B} \quad , \quad \text{midline} : y = C.$$

Since constants A, B, C are arbitrary and take all possible values giving us many different functions, we have what is called a *family of functions* $y = A\sin(Bt) + C$ and another *family of functions* $y = A\cos(Bt) + C$. The constants A, B, C are often called *parameters* of these families of functions.

Here are some important observations which will make it easier to match a given periodic function with a formula $y = A\cos(Bt) + C$ or $y = A\sin(Bt) + C$ and find the right values for the parameters A, B, and C.

- Every function in the family $y = A\cos(Bt) + C$ is at its maximum or its minimum value at $t = 0$. It is at its maximum value at $t = 0$ if $A > 0$. It is at its minimum value at $t = 0$ if $A < 0$.
- Every function in the family $y = A\sin(Bt) + C$ is at its equilibrium value at $t = 0$. If $A > 0$, the peak — the maximum value — comes first when we move toward the positive t direction. If $A < 0$, the minimum value comes first when we move toward the positive t direction. Then comes the peak.

These are important observations. Suppose that we want to match a given periodic function $f(t)$ with a formula in the family of sines $y = A\sin(Bt) + C$ or with a formula in the family of cosines $y = A\cos(Bt) + C$. To decide which family to use, we will look at the value $f(0)$ at $t = 0$. If this value is a minimum or a maximum of $f(t)$, we will look for a match in the family $y = A\cos(Bt) + C$. If the value $f(0)$ is at the equilibrium, we try to match $f(t)$ with a function in the family $y = A\sin(Bt) + C$. Also, after we find the amplitude, we will have to decide if the constant A is positive or negative; that is, if we do or do not have the reflection over the t-axis. The observations above should be helpful.

Example 8.5.1. Find the period, amplitude and midline of the following functions:

(a) $y = 4\sin\left(\frac{1}{3}t\right)$ (b) $y = -2\cos\left(\frac{\pi}{3}t\right) + 5$ (c) $y = 3\cos(\pi t) - 1$ (d) $y = -3\sin(2t) - 3$.

Solution. (a) We follow rules stated above. We have $A = 4$, $B = \frac{1}{3}$, $C = 0$. Hence, the amplitude of the function is $|4| = 4$ — the absolute value matters only if A is negative. The midline is $y = 0$. The period is:

$$\frac{2\pi}{\frac{1}{3}} = 6\pi.$$

(b) We have $A = -2$, $B = \frac{\pi}{3}$, $C = 5$. Hence, the amplitude of the function is $|-2| = 2$. The midline is $y = 5$. The period is:

$$\frac{2\pi}{\frac{\pi}{3}} = \frac{6\pi}{\pi} = 6.$$

(c) We have $A = 3$, $B = \pi$, $C = -1$. Hence, the amplitude of the function is 3. The midline is $y = -1$. The period is:

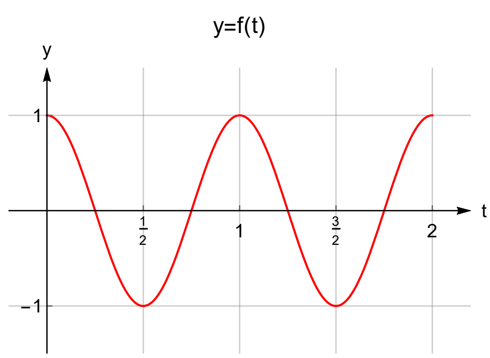
$$\frac{2\pi}{\pi} = 2.$$

(d) We have $A = -3$, $B = 2$, $C = -3$. Hence, the amplitude of the function is $|-3| = 3$. The midline is $y = -3$. The period is:

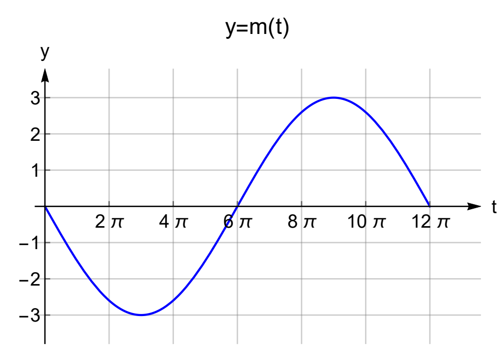
$$\frac{2\pi}{2} = \pi.$$

Example 8.5.2. In each of the graphs below you see one or two periods of a periodic function. For each of the functions find a formula in the form $y = A\sin(Bt) + C$ or $y = A\cos(Bt) + C$ that represents the graph.

(a)

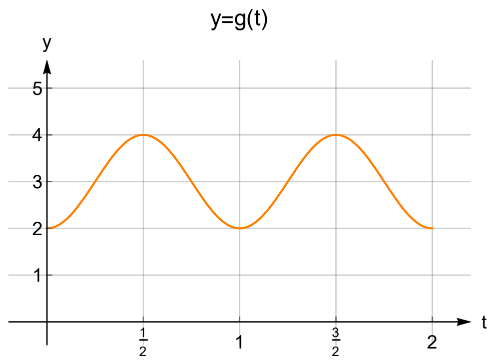


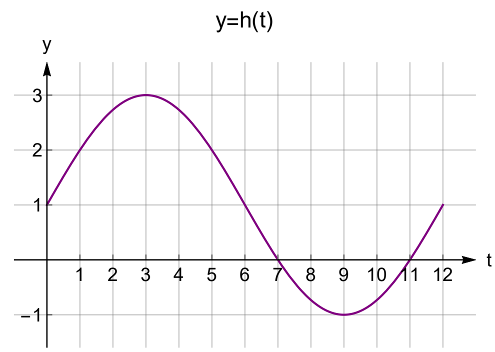
(b)



(c)

(d)





Solution. (a) Note first that at $t = 0$, $f(t)$ is at its maximum, $f(0) = 1$. Hence we will be looking for a formula for $f(t)$ in the family of functions $A\cos(Bt) + C$. To find values for the constants A, B and C, we have to find the period, amplitude and midline.

We see the same cycle repeating between $0 < t < 1$ and $1 < t < 2$. Hence we are looking at two periods of a periodic function $f(t)$ and the period of $f(t)$ is 1. The period is determined by the constant B. Since the period is 1, B must be such that:

$$\frac{2\pi}{B} = 1.$$

Solving this simple equation for B, we get $B = 2\pi$.

Hence, we know already that $f(t) = A\cos(2\pi t) + C$.

We have:

$$f_{max} = 1, \quad f_{min} = -1.$$

Hence, the amplitude is $\frac{f_{max} - f_{min}}{2} = \frac{1-(-1)}{2} = 1$. The amplitude is determined by the constant A. The amplitude 1 means that $|A| = 1$. So $A = 1$ or $A = -1$. Since $f(t)$ is at its maximum at $t = 0$ just as cosine is, we do not have a reflection over the t-axis so A is positive which gives $A = 1$. Thus, we know already that

$$f(t) = \cos(2\pi t) + C.$$

To find C, note that the equilibrium of $f(t)$ is $\frac{f_{max} + f_{min}}{2} = \frac{1+(-1)}{2} = 0$ and the midline is $y = 0$. Hence, $C = 0$ and the final formula that corresponds to the graph $y = f(t)$ is:

$$f(t) = \cos(2\pi t).$$

(b) The shape of the graph looks like that of the sine function reflected over the t-axis so we will look for a formula in the family $y = A\sin(Bt) + C$. Indeed, $m(0)$ is not at a minimum or a maximum as it would be for a function in the family of cosines.

To find B note that the period is 12π. The equation for B is then:

$$\frac{2\pi}{B} = 12\pi.$$

We divide both sides by 12π and multiply by B. We obtain:

$$B = \frac{2\pi}{12\pi} = \frac{1}{6}.$$

To find A and C observe that:

$$m_{max} = 3, \quad m_{min} = -3.$$

Hence, the amplitude is $\frac{m_{max}-m_{min}}{2} = \frac{3-(-3)}{2} = 3$, the equilibrium is $\frac{m_{max}+m_{min}}{2} = \frac{3+(-3)}{2} = 0$, the midline is $y = 0$. Hence $C = 0$. Is $A = 3$ or $A = -3$? Do we have a reflection about the t-axis? Yes. So $A = -3$ and the formula we are looking for is:

$$y = -3\sin\left(\frac{1}{6}t\right).$$

(c) Note first that at $t = 0$, $g(t)$ is at its minimum, $g(0) = 2$. Hence we will be looking for a formula for $g(t)$ in the family of functions $A\cos(Bt) + C$. To find values for the constants A, B and C, we have to find the period, amplitude and midline.

We see the same cycle repeating between $0 < t < 1$ and $1 < t < 2$. Hence we are looking at two periods of a periodic function $g(t)$ and the period of $g(t)$ is 1. The period is determined by the constant B. Since the period is 1, B must be such that:

$$\frac{2\pi}{B} = 1.$$

Solving this simple equation for B, we get $B = 2\pi$. We have:

$$g_{max} = 4, \quad g_{min} = 2.$$

Hence, the amplitude is $\frac{g_{max}-g_{min}}{2} = \frac{4-2}{2} = 1$. The amplitude 1 means that $|A| = 1$. So $A = 1$ or $A = -1$. Since $g(t)$ is at its minimum at $t = 0$ while cosine is at its maximum at $t = 0$, we have a reflection over the t-axis. So $A = -1$.

It remains to find C. We calculate the equilibrium: $\frac{g_{max}+g_{min}}{2} = \frac{4+2}{2} = 3$. Hence, $C = 3$ and the midline is $y = 3$. The formula for $g(t)$ is:

$$g(t) = -\cos(2\pi t) + 3.$$

(d) We see one cycle of the periodic function $h(t)$. $h(0)$ is not at a maximum or a minimum, so we will look for a formula in the family $y = A\sin(Bt) + C$.

The period is 12. Hence B must satisfy the equation:

$$\frac{2\pi}{B} = 12.$$

To solve the equation, we divide both sides by 12 and multiply by B:

$$\frac{2\pi}{12} = B.$$

Hence:

$$B = \frac{\pi}{6}.$$

To find A and C, observe:

$$h_{max} = 3, \quad h_{min} = -1.$$

Hence, the amplitude is:

$$\frac{h_{max} - h_{min}}{2} = \frac{3-(-1)}{2} = 2.$$

Since there is no reflection over the t-axis, $A = 2$.

To calculate C note:

$$\frac{h_{max} + h_{min}}{2} = \frac{3+(-1)}{2} = 1.$$

So $C = 1$ and the midline is $y = 1$.

Note: In each part (a), (b), (c) and (d) we solved for B the equation:

$$\frac{2\pi}{B} = period.$$

Each time we multiplied both sides by B and divided by period to obtain:

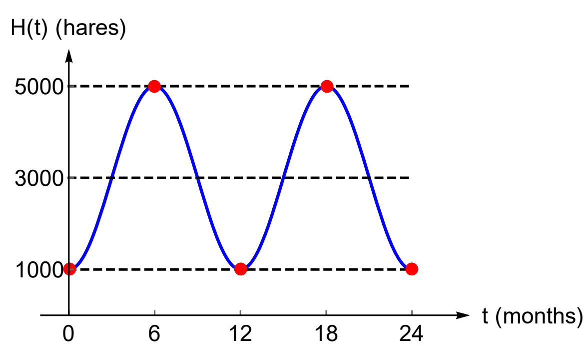
$$B = \frac{2\pi}{period}.$$

You may want to remember that given the period of a transformed sine or cosine function, we find B by dividing 2π by the period.

Modeling Periodic Processes

In this section we apply the skills of matching sine and cosine functions to modeling real-life periodic processes.

Example 8.5.3. Let's revisit the first example of this chapter, Example 8.1.1, about a population of hares, $H(t)$, in a national park. The population follows a 12-month cycle. It is at its minimum of 1000 hares at $t = 0$ which corresponds to January, and at its maximum of 5000 hares in July which corresponds to $t = 6$. Here again is the graph of $H(t)$:



Find a formula for the function $H(t)$.

Solution. We know already that the period of the function is 12, the amplitude is 2000, the midline is $H = 3000$ as $H_{max} = 5000$, $H_{min} = 1000$.

We notice that at $t = 0$, $H(t)$ is at its minimum. Hence, we will look for a formula of the form $H(t) = A\cos(Bt) + C$ with $A < 0$ as we do have a reflection about the t-axis. Since $|A|$ is equal to the amplitude and C is equal to the equilibrium:

$$A = -2000, \quad C = 3000.$$

Since the period is 12:

$$B = \frac{2\pi}{12} = \frac{\pi}{6}.$$

Thus, we can model the population of hares by the formula:

$$H(t) = -2000\cos\left(\frac{\pi}{6}t\right) + 3000.$$

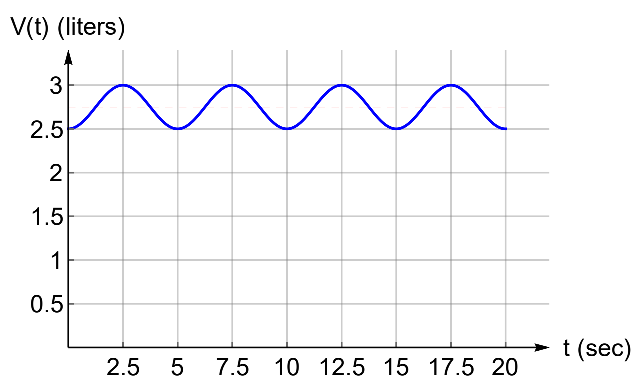
You can use your graphing calculator to graph the transformed cosine function to check that this is the function we were looking for.

Example 8.5.4. An average-sized man takes 5 seconds to breathe in and out when at rest. The volume of the air in his lungs changes as he breathes. The so-called functional residual capacity of the lungs[1] is about 2.5 liters. That is the volume of air that is always present in the lungs. The typical volume of inspiration, the tidal volume, is about 0.5 liters and so is the volume of expiration. Assume that at $t = 0$ the man finished exhaling and is about to inhale. Let $V(t)$ be the volume of air in his lungs, in liters, at time t in seconds.

(a) Sketch a rough graph of $V(t)$ in the interval $0 \leq t \leq 20$.

(b) Find the period, amplitude, midline of $V(t)$. Find a possible formula for $V(t)$.

Solution. (a) The graph of $V(t)$ looks approximately as follows:



[1] https://en.wikipedia.org/wiki/Functional_residual_capacity, accessed: 6/9/2020

At $t = 0$, the man just exhaled and is about to inhale so the volume in his lungs is at the functional residual capacity of 2.5 liters. As t increases, the man is inhaling and the volume of air in his lungs increases. The volume reaches its maximum of 3 liters mid-cycle at $t = 2.5$ seconds. Now the man begins exhaling and completes the cycle at $t = 5$ when the volume is again 2.5 liters. We are assuming here that inhaling takes the same time as exhaling so the maximum volume happens exactly in the middle of the cycle at $t = 2.5$ seconds. The same cycle repeats on every interval of the length 5 seconds.

Note: Do not confuse 2.5 liters which is the minimum volume in the man's lungs with 2.5 seconds which is the time needed to inhale or to exhale.

(b) The period — the time needed for one full cycle to be executed — is 5 seconds. We have:

$$V_{max} = 3, \quad V_{min} = 2.5.$$

Hence the amplitude is $\frac{V_{max}-V_{min}}{2} = \frac{3-2.5}{2} = 0.25$, the equilibrium is $\frac{V_{max}+V_{min}}{2} = \frac{3+2.5}{2} = 2.75$, the midline is $V = 2.75$.

To find a formula for $V(t)$, note that $V(t)$ is at its minimum at $t = 0$. Hence we will look for a transformed cosine function $V(t) = A\cos(Bt) + C$ with $A < 0$. Since the amplitude is 0.25, $A = -0.25$. C is equal to the equilibrium so $C = 2.75$. B depends on the period:

$$B = \frac{2\pi}{period} = \frac{2\pi}{5}.$$

We have a formula for $V(t)$:

$$V(t) = -0.25\cos\left(\frac{2\pi}{5}t\right) + 2.75.$$

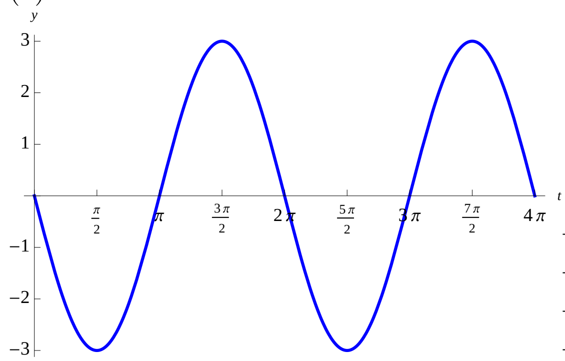
Practice Problems for Section 8.5

1. Find the amplitude, period, and midline of each of the following functions.

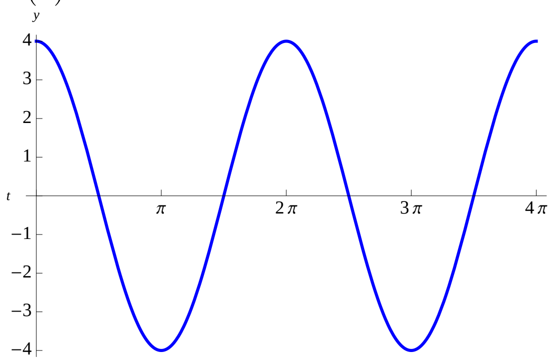
   (a) $y = \sin(4t)$
   (b) $y = -\cos(\pi t)$
   (c) $y = 16\cos\left(\frac{1}{2}t\right) + 3$
   (d) $y = 7\sin\left(\frac{\pi}{4}t\right) - 8$
   (e) $y = 1.5\sin(0.6t) - 0.1$
   (f) $y = \frac{1}{3}\cos\left(\frac{2}{\pi}t\right) + \frac{2}{5}$

2. For each of the periodic functions graphed below, find a formula in the form $y = A\sin(Bt) + C$ or $y = A\cos(Bt) + C$ that represents the graph.

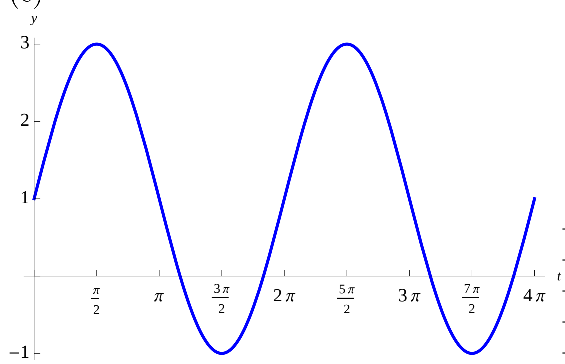
(a)



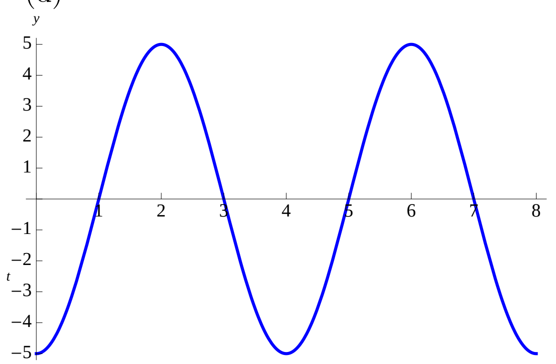
(b)



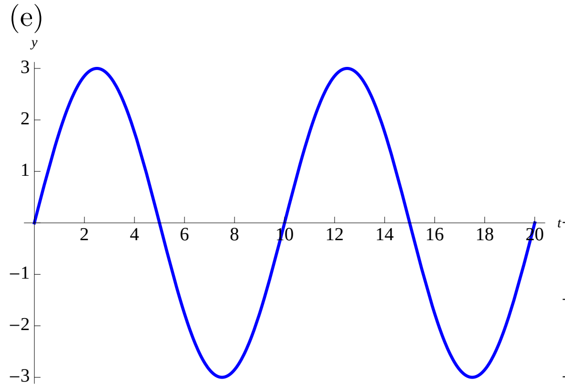
(c)



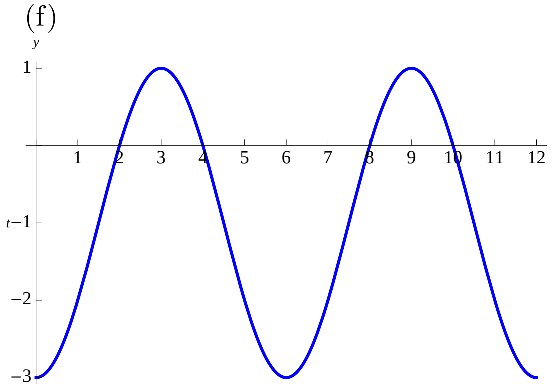
(d)



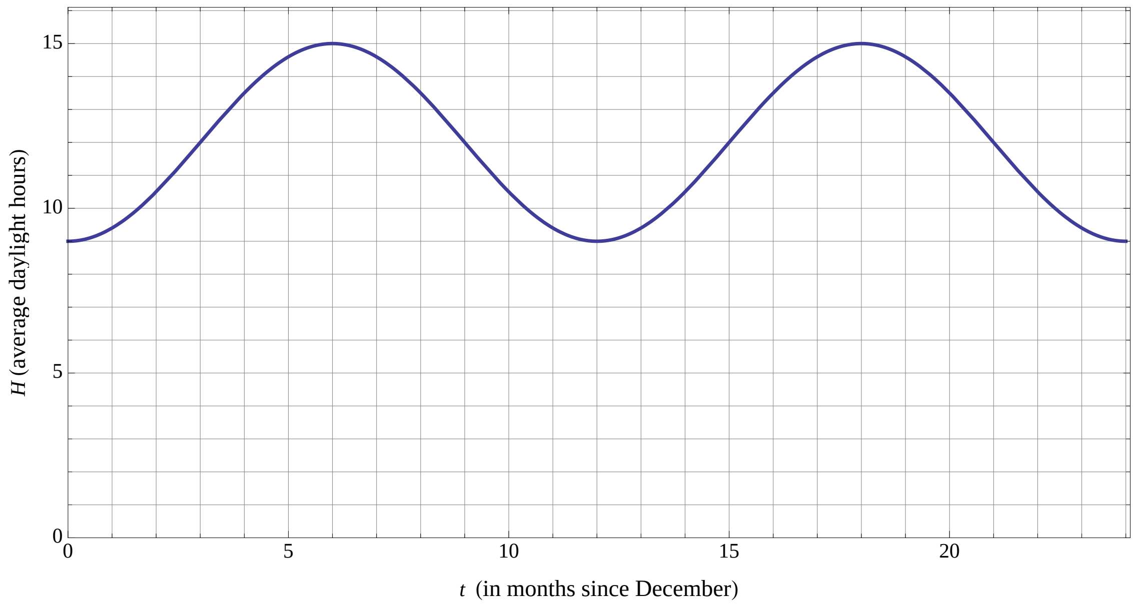
(e)



(f)



3. The average number of daylight hours H that Kingston, Rhode Island experiences is a periodic function $H = f(t)$, where $t = 0$ corresponds to the month of December. A graph modeling $f(t)$ is given below.

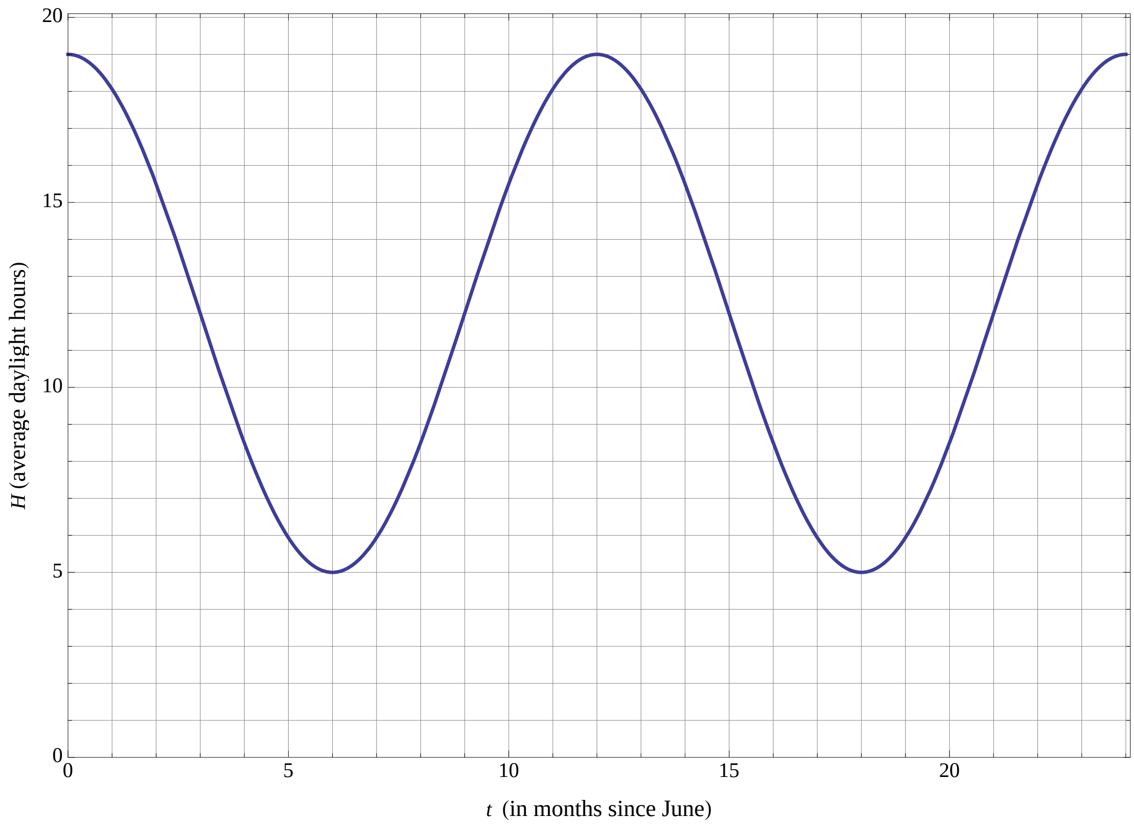


(a) Identify the amplitude, midline, and period for the periodic function modeling the average number of daylight hours in Kingston.

(b) Give a possible formula for the periodic function modeling the average number of daylight hours in Kingston.

(c) What is the average number of daylight hours in Kingston during the month of March?

4. The average number of daylight hours H that Anchorage, Alaska experiences is a periodic function $H = f(t)$, where $t = 0$ corresponds to the month of June. The graph of $f(t)$ is given below.



(a) Identify the amplitude, midline, and period for the periodic function modeling the average number of daylight hours in Anchorage.

(b) Give a possible formula for the periodic function modeling the average number of daylight hours in Anchorage.

(c) What is the average number of daylight hours in Anchorage during the month of November?

5. The number of bird species in a Rhode Island preserve oscillates between a high of 38 in June and a low of 12 in December. Write a formula for the number of bird species, N, as a function of the number of months t since December. Your answer should be of the form $N(t) = A\cos(Bt) + C$ or $N(t) = A\sin(Bt) + C$.

6. The volume of air in a the lungs of a woman at rest at certain times is shown in the following table. Assuming that the maximum volume of air in her lungs occurs at time $t = 0$ seconds and the minimum volume of air in her lungs occurs at time $t = 3$ seconds, give the formula for a periodic function $V(t)$ modeling the volume of air in the woman's lungs at any given time. Your answer should be of the form $V(t) = A\cos(Bt) + C$ or $V(t) = A\sin(Bt) + C$.

time (seconds)	0	3	6	9	12
volume (liters)	2.4	1.9	2.4	1.9	2.4

7. Each day, the tide in a harbor continuously goes in and out, raising and lowering a boat anchored there. At low tide, the boat is only 2 meters above the ocean floor. Six hours later, at peak high tide, the boat is 20 meters above the ocean floor. Six hours after peak high tide, it is low tide again. Suppose the boat is at high tide at midnight. Give a formula for a periodic function $D(t)$ modeling the boat's distance above the ocean floor as a function of time t hours since midnight. Your answer should be of the form $D(t) = A\cos(Bt) + C$ or $D(t) = A\sin(Bt) + C$.

8. You decide to ride the ferris wheel at the local carnival. You are 3 feet above the ground at the bottom of the ferris wheel and 28 feet above the ground at the top. It takes 8 seconds for the you to reach the maximum height from the minimum height and 8 seconds to reach the minimum height from the maximum height. Suppose you are at the bottom of the ride at time $t = 0$ seconds. Give a formula for a periodic function $H(t)$ modeling your height above the ground t seconds into your ferris wheel ride. Your answer should be of the form $H(t) = A\cos(Bt) + C$ or $H(t) = A\sin(Bt) + C$.

Answers to Selected Practice Problems

Section 1.1

1.

(a) t

(b) years

(c) V

(d) dollars

2.

(a) The amount of caffeine in a person's body 0 hours after drinking a cup of coffee is 96 mg.

(b) The amount of caffeine in a person's body 5 hours after drinking a cup of coffee is 48 mg.

(c) The amount of caffeine in a person's body 24 hours after drinking a cup of coffee is approximately 0 mg.

3.

(a) p; dollars

(b) C; dollars

(c) $30

4.

(a) The amount of fuel left in the fuel tank 70 miles into the drive is 6 gallons.

(b) The amount of fuel left in the fuel tank 200 miles into the drive is 1 gallon.

5.

(a) 7

(b) -8

(c) 1

6.

(a) $-9b+4$

(b) $-3b+1$

(c) $-3b$

7.

(a) 2

(b) -10

(c) $3\sqrt{2}-2$

8.

(a) h^2-2h+4

(b) h^2+2h+4

(c) b^2+2b

9.

(a) 1

(b) undefined

(c) -1

10.

(a) $\dfrac{3}{h+1}$

(b) $\dfrac{3}{h+3}$

(c) $\dfrac{3}{b+2} - 4$

11.

(a) \$16500

(b) \$9000

(c) 11 years

12.

(a) $7x + 5$

(b) $7(x + 5)$

The functions are not the same.

13.

(a) $5x^2 - 8$

(b) $(5x)^2 - 8$

(c) $5(x - 8)^2$

None of the three functions are the same.

14.

(a) all real numbers

(b) all real numbers $x \neq 1$

(c) all real numbers $x \neq -2$, $x \neq 2$

(d) all real numbers $x \geq 0$

(e) all real numbers

15.

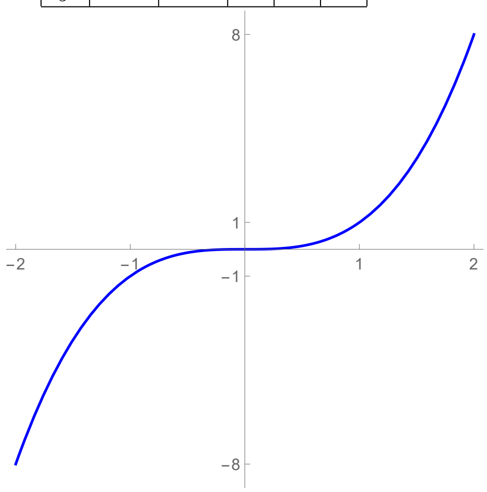
(a) S is the independent variable; D is the dependent variable

(b) $\frac{4}{5}$ of a mile

Section 1.2

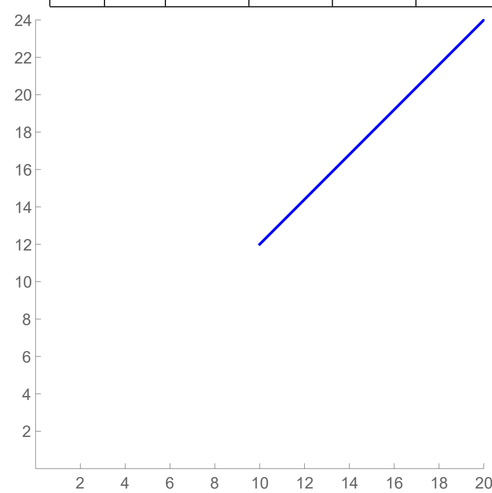
1.

x	-2	-1	0	1	2
y	-8	-1	0	1	8



2.

p	10	12	14	16	18	20
C	12	14.4	16.8	19.2	21.6	24



3.

(a) 8

(b) 10

(c) -10

4. $x \approx -2$, $x \approx 1$, and $x \approx 4$
5. $x \approx -2.3$, $x \approx 2$, $x \approx 3.4$

6.

(a) 13.2 gallons

(b) 8 gallons

(c) The gas tank is empty.

(d) 40 miles per gallon

7.

(a) 96 mg

(b) 48 mg; 24 mg

(c) decreasing

8.

(a) approximately \$1000

(b) approximately \$2000; approximately \$3800

(c) increasing

9. Not a function.

10.

(a) -4

(b) $x = -2$ and $x = 2$

(c) $x \approx -3$ and $x \approx 3$

11. increasing for $0 < t < 3$ and $9 < t < 12$; decreasing for $3 < t < 9$

Section 1.3

1.

(a) 1.5

(b) 3.7

(c) 1.5

2.

(a) $x = 0$, $x = 4$, $x = 8$

(b) $x = 2$

3. $g(x) = x^3$

4. $h(t) = -2t$; $h(5) = -10$; $h(6) = -12$

5.

t	0	1	2	3	4	5	6	7	8
$P(t)$	6.957	7.041	7.126	7.211	7.296	7.380	7.464	7.548	7.631

6. 30 miles per gallon; $G(360) = 5$, $G(420) = 3$, $G(480) = 1$

7. The population is increasing faster and faster.

8. The population is increasing faster and faster.

Section 1.4

1. horizontal intercept: 0.5; vertical intercept: -1.5

2. horizontal intercepts: -1, 1; vertical intercept: 1

3. horizontal intercept: 27; vertical intercept: 5.4

4. horizontal intercept: -2; vertical intercept: 16

5. horizontal intercept: 9/4; vertical intercept: -3

6. horizontal intercepts: none; vertical intercept: none

7. horizontal intercept: none; vertical intercept: -1

8. horizontal intercepts: -1, 2; vertical intercept: 2

9. horizontal intercepts: none; vertical intercept: 5

10. horizontal intercept: 8; vertical intercept: none

11. horizontal intercepts: 0, 4, 8; vertical intercept: 0

12. horizontal intercepts: none; vertical intercept: 3

13. vertical intercept: 300 – this is the number of feet above the lake that the ball is dropped from; horizontal intercept: $t \approx 4.33$ – this is the number of seconds that it takes after being dropped for the ball to hit the surface of the lake

14. vertical intercept: 21500 – this is the price in dollars for which the car was purchased; horizontal intercept: $t \approx 14.33$ – this is the number of years after purchase that it takes for the car's value to depreciate to $0

Section 1.5

1. 6

2. -3

3. $-\frac{1}{3}$

4. -3

5. -3

6.
   (a) 0.0845 billion people per year; 0.0835 billion people per year
   (b) No.

7.
   (a) -48 feet per second; -80 feet per second
   (b) negative; the height of the ball is decreasing as time progresses (as it was dropped)

8.

(a) -9.6 mg/hr

(b) -2.4 mg/hr

(c) The average rate of change from $t = 0$ to $t = 5$ hours is larger than the average rate of change from $t = 10$ to $t = 15$ hours. It can be seen that the rate at which the amount of caffeine in the bloodstream is changing gets smaller and smaller as time progresses.

9.

(a) positive since the graph is increasing on this interval

(b) negative since the graph is decreasing on this interval

10.

(a) -0.325 pounds/minute

(b) -0.035 pounds/minute

(c) The rate of weight loss slows as the daily amount of exercise increases.

11.

(a) approximately 2.124 million cars sold per year

(b) -3.175 million cars sold per year

(c) from 2017 to 2018 and from 2018 to 2019

Section 2.1

1. $f(x) = 3x - 6$; slope: 3; vertcal intercept: -6

2. $g(x) = 8x - 2$; slope: 8; vertical intercept: -2

3. $h(t) = 9t - 2.5$; slope: 9; vertical intercept: -2.5

4. $m(x) = 2x$; slope: 2; vertical intercept: 0

5. 2

6. $-\frac{1}{7}$

7. 26.5

8. $\frac{3}{8}$

9. 0

10. 1

11. 1.5

12. $-\frac{4}{3} \approx -1.33$

13. 2

14. 0

15. slope: 0.5; vertical intercept: 1; equation: $y = 0.5x + 1$

16. slope: $-\frac{3}{2}$; vertical intercept: -1; equation: $y = -\frac{3}{2}x - 1$

17. slope: $\frac{2}{3}$; vertical intercept: 0; equation: $y = \frac{2}{3}x$

18. slope: $-\frac{2}{3}$; vertical intercept: -1; equation: $y = -\frac{2}{3}x - 1$

19. (a) is line 5; (b) is line 2; (c) is line 1; (d) is line 4; (e) is line 3

20. $A = 1567n + 55683$

21.

(a) The initial elevation of the hiker is 1350 feet.

(b) The hiker's elevation is increasing at a rate of 35 feet per minute.

22.

(a) The initial distance from the finish line is 50 miles.

(b) The distance from the finish line is decreasing at a rate of 25 miles per hour.

Section 2.2

1. $y = -2x + 10$
2. $y = \frac{5}{2}x + 17$
3. $y = 2.1x - 0.21$
4. $y = 6x + 32$
5. $y = -\frac{1}{2}x - \frac{4}{5}$
6. $y = 3$
7. $x = 5$
8. $y = -1$
9. $f(x) = \frac{5}{2}x - 10$
10. $f(x) = -\frac{3}{2}x - \frac{1}{2}$
11. $f(x) = 2.1x - 0.21$
12. $f(x) = -x + 1$
13. $f(x) = \frac{5}{6}x + \frac{23}{6}$
14. $f(x) = 3x + 6$
15. $f(x) = x + 2$
16. $f(x) = -\frac{3}{7}x + 3$
17. (c) is not the graph of a linear function.
18.
    (a) The vertical intercept 60 is the distance in miles that the biker is from the finish line at the start of the race.

    (b) The horizontal intercept 3 is the number of hours that it takes the biker to complete the race.

    (c) The slope -20 indicates that the distance between the biker and the finish line is decreasing at a rate of 20 miles per hour.
19.
    (a) \$1900

    (b) The lamp will have a value of 4000 dollars 8.4 years after its purchase.
20.
    (a) $A(m) = 0.20m + 75$; $B(m) = 0.10m + 90$

    (b) $m = 150$

    (c) $B(m)$
21. $x \approx 1.3158$
22. $x = \frac{1}{7}$
23. No solution.
24. Infinite number of solutions.

Section 2.3

1. Linear with equation $y = -3.3x + 12.6$

2. Linear with equation $f(t) = 0.15t + 0.25$

3. Not linear.

4. Not linear.

5. Linear with equation $w(z) = -0.4z + 5.2$.

6. approximately linear with approximate equation $C = 0.555F - 17.76$

7.
   (a) $H = 2.32L + 65.53$
   (b) 172.25 cm
   (c) approximately 47.36 cm

8.
   (a) $P = 0.4d + 14.7$
   (b) 66.7 PSI

9.
   (a) $T(h) = -0.00356h + 65$
   (b) 42.89 ° F
   (c) The function $T(h)$ is decreasing.

10.
   (a) Linear; formula $H(t) = \frac{2}{3}t + 85$
   (b) The slope is 2 bpm per minute; the man's bpm is increasing at a rate of 2 bpm per minute. The vertical intercept is 85 bpm; the man's resting heart rate (heart rate when performing no exercise on the treadmill) in bpm.

Section 3.1

1. $f(x) = 4x^2 - 7x$; opens up

2. $h(t) = -5t^2 + 21$; opens down

3. $g(x) = 2x^2 - 14x$; opens up

4. $y = -4x^2 - 40x - 20$; opens down

5. horizontal intercepts: -1, 3; vertical intercept: -3; vertex: $(1, -4)$

6. no horizontal intercepts; vertical intercept: -2; vertex: $(1, -1)$

7. 0

8. $\frac{3}{2}$

9. $(6, 0)$; opens down

10.
   (a) $R(p) = p(1000 - 2p)$
   (b) \$500
   (c) The maximum revenue is \$125000. The lighting company should charge \$250 per specialty chandelier to maximize revenue.

Section 3.2

1. horizontal intercepts: -1, -2; vertical intercept: 2

2. horizontal intercepts: -2, 3 ; vertical intercept: -6

3. horizontal intercepts: -3, 2 ; vertical intercept: -6

4. horizontal intercepts: 4; vertical intercept: 16

5. horizontal intercept: -7; vertical intercept: 49

6. horizontal intercepts: $-\frac{3}{4}$, $\frac{3}{4}$; vertical intercept: -36

7. horizontal intercepts: -4, -1 ; vertical intercept: 8

8. horizontal intercepts: -1, $\frac{3}{2}$; vertical intercept: -3

9. horizontal intercepts: -8, 2; vertical intercept: 16

10. horizontal intercepts: $-\frac{1}{14}$, $\frac{1}{14}$; vertical intercept: $\frac{1}{49}$

11. horizontal intercepts: 0, 3; vertical intercept: 0

12. horizontal intercepts: 0, $\frac{2}{5}$; vertical intercept: 0

13.

    (a) $H(t) = -16t^2 + 168t$

    (b) 10.5 seconds

    (c) The maximum height that the model rocket reaches is 441 feet.

Section 3.3

1. $(3, 4)$

2. $(-2, -3)$

3. $(1, -7)$

4. $\left(-\frac{1}{2}, \frac{5}{3}\right)$

5. $f(x) = x^2 + 2x + 3$; $a = 1$, $b = 2$, $c = 3$

6. $g(x) = 5x^2 - 20x + 26$; $a = 5$, $b = -20$, $c = 26$

7. $y = -2t^2 + 16t - 27$; $a = -2$, $b = 16$, $c = -27$

8. $h(x) = -x^2 - x + \frac{17}{12}$; $a = -1$, $b = -1$, $c = \frac{17}{12}$

9. $f(x) = (x + 1)^2 - 6$; vertex: $(-1, -6)$

10. $g(x) = (x - 3)^2 + 1$; vertex $(3, 1)$

11. $h(x) = \left(x + \frac{3}{2}\right)^2 - 1$; vertex $\left(-\frac{3}{2}, -1\right)$

12. $F(x) = -(x - 5)^2 + 40$; vertex $(5, 40)$

13. $G(x) = -3(x - 1)^2 - 1$; vertex $(1, -1)$

14. $H(x) = -2(x + 4)^2 + 15$; vertex $(-4, 15)$

15. -7, 1

16. $1 - \sqrt{2} \approx -0.414$, $1 + \sqrt{2} \approx 2.414$

17. -0.5, 1.5

18. $-2 - \sqrt{10} \approx -5.162$, $-2 + \sqrt{10} \approx 1.162$

19. No real solutions.

20. $-4 - \sqrt{\frac{5}{2}} \approx -5.581$, $-4 + \sqrt{\frac{5}{2}} \approx -2.419$

Section 3.4

1. $x = -2$ and $x = 7$

2. $x = -2$ and $x = -1$

3. $x = -\frac{1}{2} = -0.5$ and $x = \frac{1}{3} \approx 0.333$

4. $x = 1 - \sqrt{3} \approx -0.732$ and $x = 1 + \sqrt{3} \approx 2.732$

5. $x = -3$ and $x = -\frac{5}{4} = 1.25$

6. $x = \dfrac{1 - \sqrt{41}}{2} \approx -2.701$ and $x = \dfrac{1 + \sqrt{41}}{2} \approx 3.702$

7. horizontal intercepts: -3, 1; vertical intercept: -3; vertex: $(-1, -4)$

8. horizontal intercepts: $-4 - \sqrt{6} \approx -6.450$, $-4 + \sqrt{6} \approx -1.551$; vertical intercept: 10; vertex: $(-4, -6)$

9. horizontal intercepts: $\frac{3-2\sqrt{3}}{3} \approx -0.155$, $\frac{3+2\sqrt{3}}{3} \approx 2.155$; vertical intercept: -1; vertex: $(1, -4)$

10. no horizontal intercepts; vertical intercept: 5; vertex $(2, 1)$

11. 0; one real solution

12. 132; two distinct real solutions

13. -116; no real solutions

14. 56; two distinct real solutions

15.
   (a) $H(t) = -16t^2 + 50t + 6$
   (b) The ball reaches its maximum height approximately 1.56 seconds after it is thrown straight up and that maximum height is approximately 45.06 feet.

16. $f(x) = 3(x + 1)(x - 2)$

17. $g(x) = \left(x - (-3 - \sqrt{2})\right)\left(x - (-3 + \sqrt{2})\right)$

18. $f(x) = -2(x + 1)(x - 2)$; $f(x) = -2x^2 + 2x + 4$

19. $g(x) = (x - 2)(x + 2)$; $g(x) = x^2 - 4$

Section 4.1

1. 2^{-2}

2. 2^{-6}

3. 2^5

4. 2^{-12}

5. 5^{-1}

6. 5^{-3}

7. 5^2

8. 5^2

9. 1

10. x^4y^2

11. Cannot be simplified.

12. $8x^6y^{12}$

13. Cannot be simplified.

14. $\pi^2x^{20}y^8$

15. Cannot be simplified.

Section 4.2

1. $x = -\sqrt[4]{20} \approx -2.115$ and $x = \sqrt[4]{20} \approx 2.115$
2. $b = -\sqrt[30]{\frac{34}{3}} \approx -1.084$ and $b = \sqrt[30]{\frac{34}{3}} \approx 1.084$
3. $x = \sqrt[3]{-\frac{2}{5}} \approx -0.737$
4. No real solutions.
5. $x = 1 - \sqrt{3} \approx -0.732$ and $x = 1 + \sqrt{3} \approx 2.732$
6. $x = -4$
7. $3^{5/2}$
8. $3^{5/2}$
9. $3^{13/6}$
10. $3^{-1/20}$
11. Cannot be simplified.
12. $xy\sqrt[3]{x}$
13. $\dfrac{y^2}{\sqrt[3]{x}}$
14. Cannot be simplified.

Section 4.3

1. $y = 4x^3$; $k = 4$; $p = 3$
2. $y = 3x^3$; $k = 3$; $p = 3$
3. Not a power function.
4. $y = 0.1x^3$; $k = 0.1$; $p = 3$
5. Not a power function.
6. $y = 9x^2$; $k = 9$; $p = 2$
7. Yes; $k = \frac{0.034}{\mu}$; $p = 2$
8. 59.5 feet; 238 feet
9. Factor of 4.
10. 416.5 feet
11. p is even; k is positive
12. p is odd; k is positive
13. p is odd; k is negative
14. p is even; k is negative
15. (B)
16. (B)
17. A is $y = \frac{1}{4}x^4$; B is $y = x^4$; C is $y = -x^4$
18. 4.298 cm

Section 4.4

1. $y = 10x^{1/2}$; $k = 10$; $p = \frac{1}{2}$
2. $y = 0.1x$; $k = 0.1$; $p = 1$
3. $y = \frac{1}{4}x^{17/12}$; $k = \frac{1}{4}$; $p = \frac{17}{12}$
4. Not a power function.
5. $y = \frac{1}{3}x^2$; $k = \frac{1}{3}$; $p = 2$
6. Not a power function.
7. p is odd; k is positive
8. p is even; k is negative

9. p is odd; k is negative

10. p is even; k is positive

11. (C)

12. (A)

13. (A) is $y = x^{\frac{1}{2}}$; (B) is $y = 2x^{\frac{1}{2}}$; (C) is $y = -x^{\frac{1}{2}}$

14. $f(x) = 3x^{\frac{1}{2}}$

15. $g(x) = 2x^{-1}$

16. yes; $T = \frac{2\pi}{\sqrt{g}} L^{\frac{1}{2}}$ where the coefficient is $\frac{2\pi}{\sqrt{g}}$ and the exponent is $\frac{1}{2}$

17. approximately 1.42 seconds

18. approximately 0.47 meters/sec^2

Section 5.1

1. exponential growth; intial value: 120; growth factor: 1.7

2. exponential decay; intial value: 1500; growth factor: 0.95

3. exponential decay; intial value: 60; growth factor: 0.83

4. exponential growth; intial value: 130; growth factor: 1.02

5.

   (a) $B(t) = 2000(1.035)^t$; exponential function

   (b) \$2544.56

   (c) growth factor: 1.035; growth rate: 3.5%

6.

   (a) $V(t) = 500t + 7000$; linear function

   (b) 500 dollars per year

   (c) \$13000

7.

   (a) $V(t) = 7000(1.095)^t$; exponential function

   (b) growth factor: 1.095; growth rate: 9.5%

   (c) \$20800.20

8.

   (a) $P(t) = 750t + 12000$; linear function

   (b) 750 people per year

9.

   (a) $P(t) = 12000(1.107)^t$; exponential function

   (b) growth factor: 1.107; growth rate: 10.7%

10.

   (a) nutrient b; 15%

   (b) nutrient c; -25%

11.

   (a) $M(t) = 40(0.963)^t$; growth factor: 0.963

   (b) approximately 27.44 mg

12.

   (a) 0.9772

   (b) -2.28%

   (c) approximately 12.52 mg

13. $y = 30(0.64)^t$; initial value: 30; growth factor: 0.64; decreasing

14. $y = 12.6491(1.09)^t$; initial value: 12.6491; growth factor: 1.09; increasing

15. Not an exponential function.

16. $y = 70(1.1487)^t$; initial value: 70; growth factor: 1.1487; increasing

17. $y = 2.5(2)^t$; initial value: 2.5; growth factor: 2; increasing

18. $y = 60(0.7937)^t$; initial value: 60; growth factor: 0.7937; decreasing

Section 5.2

1.

x	-3	-2	-1	0	1	2	3
$y = 3^x$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27

2.

x	-3	-2	-1	0	1	2	3
$y = \left(\frac{1}{3}\right)^x$	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

3. $A = 100$; $b < 1$

4. Graph (C)

5. Graph (B)

6. $y = 10(1.2)^t$

7. $y = 200(0.8)^t$

8. $y = 500(0.7)^x$

9. $y = 30(1.6)^x$

10.

(a) $M(t) = 64(0.5)^t$

(b)

t (days)	0	1	2	3	4
$M(t)$ mg	64	32	16	8	4

11.

(b) Yes; $C(t) = 100(2)^t$

Section 5.3

1. possible exponential function; $f(t) = 5000(0.8)^t$; initial value: 5000; growth factor: 0.8

2. possible exponential function; $g(t) = 32(1.5)^t$; initial value: 32; growth factor 1.5

3. not an exponential function

4. possible exponential function; $h(t) = 1024(1.25)^t$; initial value: 1024; growth factor: 1.25

5. not an exponential function

6. possible exponential function; $z(t) = 250(\sqrt{2})^t \approx 250(1.4142)^t$

7. possible exponential function; $m(t) = 64\left(\sqrt[3]{\frac{1}{2}}\right)^t \approx 64(0.7937)^t$

8. $y = 500(1.8)^x$; $y(4) = 5248.8$

9. $y = -80x + 300$; $y(4) = -20$

10.

(a) Yes.

(b) 760 mmHg

(c) $P(H) = 760(0.887)^H$

(d) approximately 263.05 mmHg

Section 5.4

1. initial amount: 47; doubling time: 5
2. initial amount: 800; doubling time: 9
3. initial amount: 89; doubling time: 3
4. initial amount: 120; doubling time: 4
5. initial amount: 75; half-life: 7
6. initial amount: 920; half-life: 10
7. initial amount: 875; half-life: 3
8. initial amount: 63; half-life: 5
9. $f(t) = 1500(2)^{\frac{t}{8}} = 1500(1.0905)^t$
10. $f(t) = 750(3)^{\frac{t}{9}} = 750(1.1298)^t$
11. $f(t) = 50\left(\frac{1}{2}\right)^{\frac{t}{11}} = 50(0.9389)^t$
12. $f(t) = 60\left(\frac{1}{3}\right)^{\frac{t}{5}} = 60(0.8027)^t$
13. doubling time: 3
14. half-life: approximately 2
15.
    (a) 2000 bacteria
    (b) 20 minutes
    (c) $P(t) = 2000(2)^{\frac{t}{20}} \approx 2000(1.0353)^t$; growth rate: 3.53%
16.
    (a) 96 mg
    (b) 5 hours
    (c) $C(t) = 96\left(\frac{1}{2}\right)^{\frac{t}{5}} \approx 96(0.8706)^t$; growth rate: −12.94%

Section 5.5

1. 7.389
2. 4.482
3. 0.368
4. 2.854
5. (i) is graph C; (ii) is graph B; (iii) is graph A
6. increasing; continous growth rate: 12%
7. decreasing; continous growth rate: −2.9%
8. increasing; continous growth rate: 6.5%
9. decreasing; continous growth rate: −38%
10. $y = 120(1.3771)^t$
11. $y = 1500(0.9066)^t$
12. $y = 200(0.8521)^t$
13. $y = 1700(1.0942)^t$
14.
    (a) $P(t) = 30e^{-0.033t}$
    (b) approximately 16.56 mg
15. $I(t) = 12e^{-0.0533t}$; approximately 3.33 μg

Section 6.1

1. -6
2. undefined
3. undefined
4. 3
5. -2
6. $-\frac{1}{2}$
7. 24
8. 0
9. 0
10. undefined
11. $\frac{7}{2}$
12. -4
13. $-\frac{7}{6}$
14. undefined
15. -12
16. undefined
17. 5
18. 4
19. approximately -1.6
20. approximately 0.2
21. approximately -0.7
22. approximately 0.3
23. 0.8451
24. 2.1197
25. 0.4135
26. 24.9334

Section 6.2

1. $\log(5) + \log(z) - \log(x)$
2. $5\log(x) + 3\log(y) + 2\log(z)$
3. $\log(y) + \log(z) - \log(6) - \log(x)$
4. $2\log(y) + 3\log(z) - \frac{1}{2}\log(4) - \frac{1}{2}\log(x)$
5. Cannot simplify.
6. Cannot simplify.
7. $2 + \log(y) + \frac{1}{2}\log(x)$
8. $1 + \log(x) - \log(z) - \log(y)$
9. $2\ln(z) - \ln(2) - \ln(x)$
10. $\frac{3}{2}\ln(x) + \frac{3}{2}\ln(y) + \frac{1}{2}\ln(z)$
11. Cannot simplify.
12. $1 + \ln(x) + \ln(y) - \ln(3) - \ln(z)$
13. $2 + \ln(y) - 3\ln(z)$
14. $\ln(x) - \ln(3) - 1$
15. $-3\ln(x) - \ln(y)$
16. $\frac{1}{2} - \ln(x) - \ln(z)$
17. Not possible.
18. $\log\left(\frac{A^2}{B^3}\right)$
19. $\ln\left(B^{-3}\sqrt{AB}\right)$
20. Not possible.

21. $\ln\left(\frac{\sqrt[3]{A}B^2}{10}\right)$

22. Not possible.

23. $x = \dfrac{\log(500)}{\log(5)} \approx 3.861$

24. $t = \dfrac{\log(2)}{\log(1.15)} \approx 4.959$

25. $t = \dfrac{\log\left(\frac{1}{2}\right)}{\log(0.81)} \approx 3.289$

26. $x = \dfrac{\log\left(\frac{4}{3}\right)}{\log\left(\frac{7}{5}\right)} \approx 0.855$

27. $t = \dfrac{10\log\left(\frac{4}{9}\right)}{\log(0.5)} \approx 11.69$

28. $t = -\dfrac{\ln\left(\frac{10}{49}\right)}{0.098} \approx 16.217$

29. $\dfrac{\ln(8)}{\ln(5)} \approx 1.292$

30. $\dfrac{\ln(12)}{\ln(3)} \approx 2.262$

Section 6.3

1. $y = 208e^{-0.1625t}$

2. $y = e^{0.0862t}$

3. $y = 700e^{0.2776t}$

4. $y = 2000e^{-0.0943t}$

5.
   (a) approximately 11.36 years
   (b) approximately 18.01 years

6.
   (a) $S(t) = S_0e^{0.0231t}$
   (b) approximately 47.6 minutes; no

7.
   (a) $G(t) = G_0e^{0.0277t}$
   (b) approximately 14.64 hours

8.
   (a) $N(t) = 2e^{-0.3466t}$
   (b) approximately 3.03 hours

9.
   (a) $C(t) = 96e^{-0.1386t}$
   (b) approximately 5 hours
   (c) approximately 8.69 hours; no
   (d) approximately 16.32 hours; yes

10.
   (a) $V(t) = 7000(1.15)^t$
   (b) approximately 5.9 years

11. $0.001\ \dfrac{\text{mol}}{\text{L}}$

12. The pH is 4.

Section 7.1

1. $g(x) = f(x) - 7$

2. $g(x) = f(x) + 1.5$

3. $g(x) = f(x - 2)$

4. $g(x) = f(x + 3)$

5. $g(x) = f(x - 6) + 2.3$

6. $g(x) = f(x - 0.5) - 1.63$

7. $g(x) = f(x - 10) + 12$

8. $g(x) = f(x + 7.1) - 3.2$

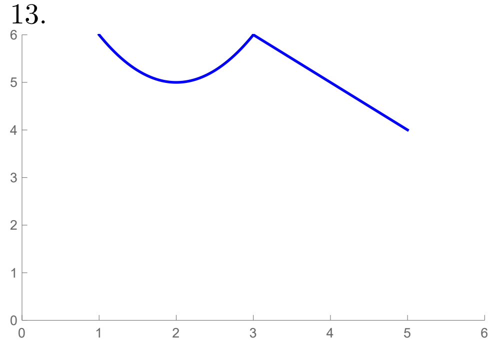
9. The graph of $f(x)$ is the graph of $y = x^2$ shifted left 4 units.

10. The graph of $g(t)$ is the graph of $y = e^t$ shifted down 3.2 units.

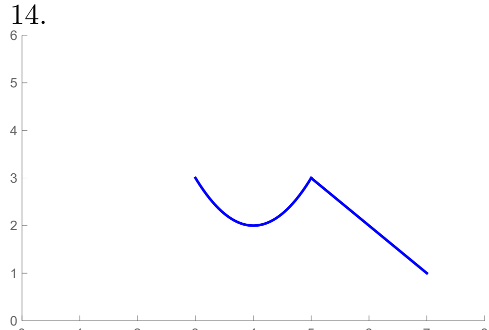
11. The graph of $h(w)$ is the graph of $y = \sqrt{w}$ shifted right 4.1 units and up 5.2 units.

12. The graph of y is the graph of $f(t) = (0.9)^t$ shifted left 1 unit and down 3 units.

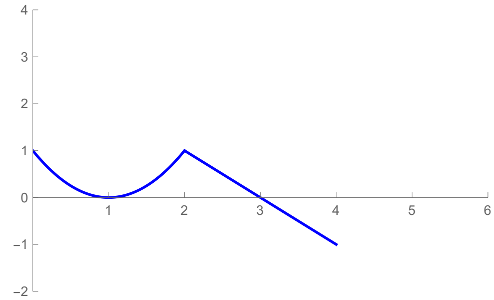
13.



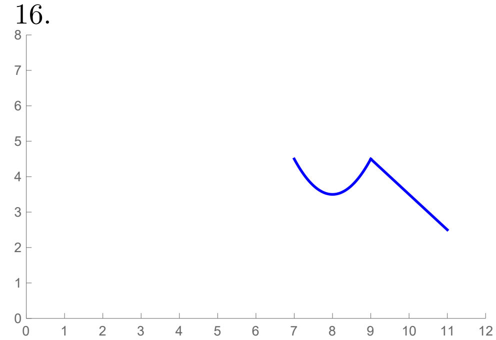
14.



15.



16.



17.

t	-3	2	7	12
y	0.1	0.2	0.4	0.8

18.

t	0	5	10	15
y	-1.9	-1.8	-1.6	-1.2

19.

t	2	7	12	17
y	6.1	6.2	6.4	6.8

20.

t	-3.1	1.9	6.9	11.9
y	1.2	1.3	1.5	1.9

21.

(a) $g(t) = f(t + 15)$

(b) $h(t) = f(t) + 5$

22. $g(t) = C(t) + 2.3$

Section 7.2

1. $g(x) = 3f(x)$

2. $g(x) = \frac{1}{20}f(x)$

3. $g(x) = -3f(x)$

4. $g(x) = \frac{1}{4}f(x-2)$

5. $g(x) = \frac{1}{5}f(-x) + 3$

6. $g(x) = f\left(\frac{1}{7}x\right)$

7. $g(x) = f(2x)$

8. $g(x) = 6f\left(-\frac{1}{5}(x-2)\right) + 10$

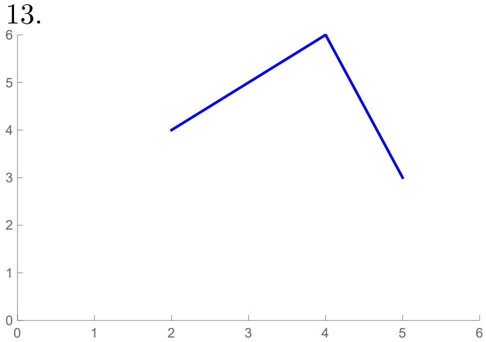
9. The graph of $f(x)$ is the graph of $y = x^3$ shifted left 4 units, scaled vertically by a factor of 2, and reflected over the x-axis

10. The graph of $g(t)$ is the graph of $y = \sqrt[3]{t}$ shifted right 5 units, compressed horizontally by a factor of 2, and shifted up 9 units.

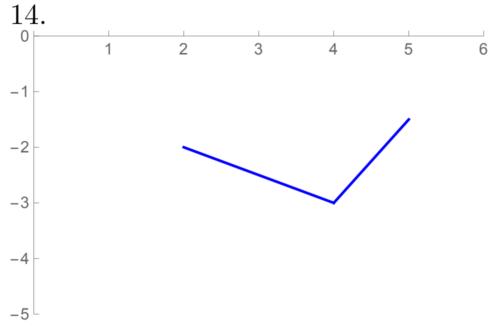
11. The graph of $h(w)$ is the graph of $y = e^w$ compressed horizontally by a factor of 2, reflected over the y-axis, compressed vertically by a factor of 4, and shifted down 7 units.

12. The graph of y is the graph of $f(x) = 2^x$ shifted left 1 unit, compressed horizontally by a factor of 7, and shifted down 3 units.

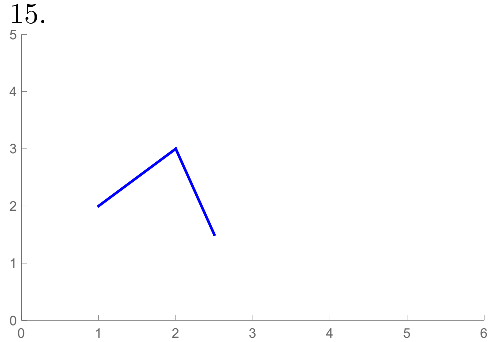
13.



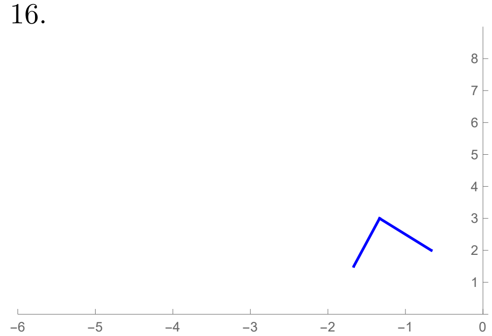
14.



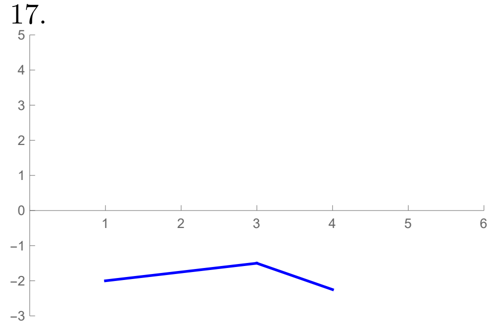
15.



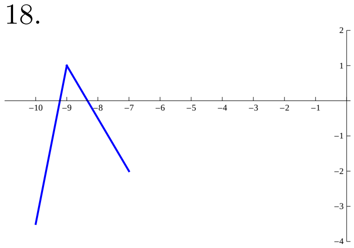
16.



17.



18.



19.

x	0	2	4	6
y	4	−1	−12	20

20.

x	0	2	4	6
y	−1	0.25	3	−5

21.

x	−1	1	3	5
y	1	−4	−15	17

22.

x	−3	−2	−1	0
y	−29	19	2.5	−5

Section 7.3

1. $f(g(x)) = 2x + 1$; $g(f(x)) = 2x + 2$
2. $f(g(x)) = -\frac{3}{2}x + 14$; $g(f(x)) = -\frac{3}{2}x + \frac{11}{2}$
3. $f(g(x)) = 15x^2 - 35$; $g(f(x)) = 75x^2 - 300x + 295$
4. $f(g(x)) = 64x^2 - 32x - 4$; $g(f(x)) = 8x^2 - 48x + 9$
5. $f(g(x)) = \sqrt{4 - x}$; $g(f(x)) = 4 - \sqrt{x}$
6. $f(g(x)) = \sqrt{\sqrt{x+7} - 7}$; $g(f(x)) = \sqrt{\sqrt{x-7} + 7}$
7. $f(g(x)) = \dfrac{4e^x + 1}{5}$; $g(f(x)) = 2e^{\frac{2x+1}{5}}$
8. $f(g(x)) = 5^{2x-1}$; $g(f(x)) = 2(5^x) - 1$
9. $f(g(x)) = \dfrac{3x+1}{(3x+1)^2 - 9}$; $g(f(x)) = 3\left(\dfrac{x}{x^2 - 9}\right) + 1$
10. $f(g(x)) = \frac{5}{2x} - 3$; $g(f(x)) = \frac{x+1}{3x-7}$
11. 23
12. 7
13. $\frac{5}{4}$
14. 2
15. $f(x) = 2x^5 + 1$; $g(x) = x - 5$
16. $f(x) = \sqrt[3]{x}$; $g(x) = 7 - x$
17. $f(x) = 2^x$; $g(x) = \sqrt{x}$
18. $f(x) = \frac{1}{x}$; $g(x) = 5x - 7$
19. $f(x) = \ln(x)$; $g(x) = 16x + 5x^2$
20. $f(x) = x^3$; $g(x) = \dfrac{2x+1}{x-1}$
21. $f(x) = e^x$; $g(x) = 3x + 2$
22. $f(x) = e^x$; $g(x) = \sqrt{x - 5}$

Section 7.4

1. Not inverses.
2. Not inverses.
3. Inverses.
4. Not Inverses.
5. Inverses.
6. Inverses.

7. $f^{-1}(x) = \frac{x}{5}$

8. $f^{-1}(t) = \frac{2}{t}$

9. $f^{-1}(x) = -2x + 6$

10. No inverse function.

11. $f^{-1}(x) = \sqrt[3]{\dfrac{10 - x}{3}}$

12. $f^{-1}(x) = x^3 - 1$

13. No inverse function.

14. $f^{-1}(x) = \dfrac{x + 1}{3x}$

15. $f^{-1}(x) = \dfrac{-x - 5}{3x - 2}$

16. $f^{-1}(x) = \log_2(x) = \dfrac{\log(x)}{\log(2)} = \dfrac{\ln(x)}{\ln(2)}$

17. $f^{-1}(x) = \ln\left(\dfrac{x + 6}{5}\right)$

18. $f^{-1}(x) = e^x - 1$

19. $f^{-1}(x) = 6^{2x+20}$

20. $f^{-1}(x) = \frac{5}{9}x - \frac{160}{9}$

Section 8.1

1. (a), (c), and (f) are periodic;
For (a) amplitude≈0.65; midline y ≈0.35; period≈3
For (c) amplitude≈1.4; midline y ≈0; period≈6.3
For (f) amplitude≈3; midline y ≈1; period≈1.6

2. (b) period: 12; amplitude: 9; midline: $y = 11$

3. period: 16; amplitude: 12.5; midline: $y = 15.5$

Section 8.2

1. $\frac{3\pi}{20} \approx 0.471$ radians; Quadrant I

2. $-\frac{\pi}{18} \approx -0.175$ radians; Quadrant IV

3. $\frac{10\pi}{3} \approx 10.472$ radians; Quadrant III

4. $-\frac{281\pi}{90} \approx -9.809$ radians; Quadrant II

5. $10°$; Quadrant I

6. $-18°$; Quadrant IV

7. $\frac{1440}{\pi} \approx 458.366°$; Quadrant II

8. $216°$; Quadrant III

9. $3\pi \approx 9.425$ inches

10. $\frac{10\pi}{3} \approx 10.472$ centimeters

Section 8.3

1. $-\frac{\sqrt{2}}{2}$

2. $-\frac{\sqrt{2}}{2}$

3. $-\frac{1}{2}$

4. $\frac{\sqrt{3}}{2}$

5. -1

6. 1

7. $\frac{\sqrt{3}}{2}$

8. $-\frac{1}{2}$

9. $-\frac{\sqrt{3}}{2}$

10. $-\frac{1}{2}$

Section 8.4

1. period: 2π; amplitude: 5; midline: $y = 0$
y is the graph of $f(t) = \cos(t)$ stretched vertically by a factor of 5

2.period: 2π; amplitude: 4; midline: $y = 0$
y is the graph of $f(t) = \sin(t)$ stretched vertically by a factor of 4 and reflected over the t-axis

3. period $\frac{2\pi}{5}$: ; amplitude: 1; midline: $y = 0$
y is the graph of $f(t) = \sin(t)$ compressed horizontally by a factor of 5

4. period: 6; amplitude: 1; midline: $y = 0$
y is the graph of $f(t) = \cos(t)$ compressed horizontally by a factor of π and stretched horizontally by a factor of 3

5. period: 2π; amplitude: 1; midline: $y = 10$
y is the graph of $f(t) = \cos(t)$ shifted up 10 units

6. period: 2π; amplitude: 1; midline $y = -6$:
y is the graph of $f(t) = \sin(t)$ shifted down 6 units

7. period: $\frac{10\pi}{3}$; amplitude: 1; midline: $y = 0$
y is the graph of $f(t) = \sin(t)$ compressed horizontally by a factor of 3 and stretched horizontally by a factor of 5

8. period: 2π; amplitude: 2; midline: $y = -6$
y is the graph of $f(t) = \cos(t)$ stretched vertically by a factor of 2 and shifted down 6 units

9. period π: ; amplitude: 3; midline: $y = 0$
y is the graph of $f(t) = \sin(t)$ compressed horizontally by a factor of 2, stretched vertically by a factor of 3, and reflected over the t-axis

10. period: 4; amplitude: 10; midline: $y = 4$
y is the graph of $f(t) = \sin(t)$ compressed horizontally by a factor of π, stretched horizontally by a factor of 2, stretched vertically by a factor of 10, and shifted up 4 units

Section 8.5

1.

(a) amplitude: 1; period: $\frac{\pi}{2}$; midline: $y = 0$
(b) amplitude: 1; period: 2; midline: $y = 0$
(c) amplitude: 16; period: 4π ; midline: $y = 3$
(d) amplitude: 7; period: 8; midline: $y = -8$
(e) amplitude: 1.5; period: $\frac{10\pi}{3}$; midline: $y = -0.1$
(f) amplitude: $\frac{1}{3}$; period: π^2; midline: $y = \frac{2}{5}$

2.

(a) $y = -3\sin(t)$
(b) $y = 4\cos(t)$
(c) $y = 2\sin(t) + 1$
(d) $y = -5\cos\left(\frac{\pi}{2}t\right)$
(e) $y = 3\sin\left(\frac{\pi}{5}t\right)$
(f) $y = -2\cos\left(\frac{\pi}{3}t\right) - 1$

3.

(a) amplitude: 3; midline: $y = 12$; period: 12

(b) $f(t) = -3\cos\left(\frac{\pi}{6}t\right) + 12$

(c) 12 hours

4.

(a) amplitude: 7; midline: $y = 12$; period: 12

(b) $f(t) = 7\cos\left(\frac{\pi}{6}t\right) + 12$

(c) approximately 6 hours

5. $N(t) = -13\cos\left(\frac{\pi}{6}t\right) + 25$

6. $V(t) = 0.25\cos\left(\frac{\pi}{3}t\right) + 2.15$

7. $D(t) = 9\cos\left(\frac{\pi}{6}t\right) + 11$

8. $H(t) = -12.5\cos\left(\frac{\pi}{8}t\right) + 15.5$