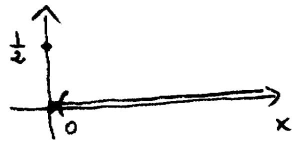
MTH 436 Spring 03 Homework 7 – Solutions

1) Let $x = 0$. We have

$$f_n(0) = \frac{1}{2} \quad \text{at } n = 1, 2, \ldots.$$

Thus $f_n(0) \to \frac{1}{2}$ as $n \to +\infty$. Let $x \neq 0$. Then $nx \to +\infty$ as $n \to +\infty$. Hence, $f_n(x) \to 0$ as $n \to +\infty$. The pointwise limit of $\{f_n\}_{n=1}^{\infty}$ is:

$$f(x) = \begin{cases} \frac{1}{2}, & x = 0 \\ 0, & x > 0. \end{cases}$$



The sequence does not converge uniformly in $[0, +\infty)$. Indeed, all functions f_n are continuous in $[0, +\infty)$, while f is not.

2) For every fixed $x \in (-\infty, +\infty)$, $f_n(x) \to 0$ as $n \to +\infty$. Thus $f(x) \equiv 0$ is the pointwise limit of $\{f_n\}_{n=1}^{\infty}$ on $[0, +\infty)$. To show that the convergence is uniform, let's find $f_n'(x)$ and determine the behavior of f_n on $(-\infty, +\infty)$.

$$f_n'(x) = \frac{1 - n^2x^2}{(1 + n^2x^2)^2}.$$

For each fixed $n \in \mathbb{N}$, f_n has two critical points $a_n = -\frac{1}{n}$, $b_n = \frac{1}{n}$. f_n' is negative in $(-\infty, -\frac{1}{n})$, positive in $(-\frac{1}{n}, \frac{1}{n})$, and negative in $(\frac{1}{n}, +\infty)$. We notice that f_n is odd and $f_n(x) > 0$ for $x > 0$, $f_n(x) < 0$ for $x < 0$. We conclude that f_n reaches its global maximum at b_n and its global minimum at a_n. Thus:

$$-\frac{1}{2n} = f_n(a_n) \leq f_n(x) \leq f_n(b_n) = \frac{1}{2n}.$$

To prove that $f_n \rightrightarrows f$, take $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $N > \frac{1}{2\varepsilon}$. Then for all $x \in (-\infty, +\infty)$ and $n \geq N$ we have

$$|f_n(x) - f(x)| = |f_n(x)| \leq \frac{1}{2n} < \varepsilon.$$

We conclude that $f_n \rightrightarrows f$ on $(-\infty, +\infty)$.

3) Notice that for each fixed $x > 0$

$$g_n(x) = \frac{1}{x}$$

for all $n \geq \frac{1}{x}$. Thus, for any $x > 0$, $g_n(x) \to \frac{1}{x}$ as $n \to +\infty$. For $x = 0$, $g_n(0) = 0 \to 0$ as $n \to +\infty$. Therefore, $g_n \to g$ on $[0,1]$, where

$$g(x) = \begin{cases} \frac{1}{x}, & x \in (0,1] \\ 0, & x = 0. \end{cases}$$

The convergence is not uniform as all g_n are continuous in $[0,1]$ and g is not.

4) $f_n \rightrightarrows f$ on D if and only if

$$\underset{\varepsilon>0}{\forall} \ \underset{N \in \mathbb{N}}{\exists} \ \underset{n \geq N}{\forall} \ \underset{x \in D}{\forall} \ |f_n(x) - f(x)| < \varepsilon.$$

The latter condition is equivalent to the condition:

$$\underset{\varepsilon>0}{\forall} \ \underset{N \in \mathbb{N}}{\exists} \ \underset{n \geq N}{\forall} \ \underset{x \in D}{\forall} \ |f_n(x) - f(x)| \leq \varepsilon. \qquad (1)$$

Observe that, for any given $\varepsilon > 0$ and $n \in \mathbb{N}$, the condition

$$M_n = \sup_{x \in D} |f_n(x) - f(x)| \leq \varepsilon.$$

is equivalent to

$$\underset{x \in D}{\forall} \ |f_n(x) - f(x)| \leq \varepsilon.$$

Therefore, (1) is equivalent to

$$\forall_{\varepsilon>0} \ \exists_{N\in\mathbb{N}} \ \forall_{n\geq N} \quad M_n \leq \varepsilon . \qquad (2)$$

Given that $M_n \geq 0$ for $n=1,2,\ldots$, (2) is equivalent to $\lim_{n\to+\infty} M_n = 0$. Thus, $f_n \rightrightarrows f$ on D if and only if $M_n \to 0$.

(Note: Observe that nowhere in this proof we claimed that for <u>a given</u> $\varepsilon>0$, $|f_n(x)-f(x)|<\varepsilon$ is equivalent to $|f_n(x)-f(x)|\leq\varepsilon$.)

5) Suppose that f is not bounded on $[a,b]$. Let $N\in\mathbb{N}$ be such that

$$|f_n(x)-f(x)|<1 \text{ for all } n\geq N \text{ and all } x\in[a,b]. \quad (3)$$

(Such N exists as $f_n \rightrightarrows f$ on $[a,b]$.) (3) implies that

$$|f_n(x)| \geq |f(x)|-1 \quad \text{for all } x\in[a,b],\ n\geq N.$$

In particular, for $n=N$, we have

$$|f_N(x)| \geq |f(x)|-1 \quad \text{for all } x\in[a,b]. \quad (4)$$

Let $K\in\mathbb{N}$ be such that $K-1>K_N$. Since f is not bounded, there exists $x_K\in[a,b]$ such that

$$|f(x_K)|>K.$$

(4) and the latter inequality imply that

$$|f_N(x_K)| \geq K-1 > K_N .$$

Contradiction. Thus, f is bounded on $[a,b]$.

Uniform convergence cannot be replaced by pointwise convergence. Indeed in Problem 3, g_n are bounded in $[0,1]$, $g_n \to g$ on $[0,1]$, yet g is not bounded in $[0,1]$