MTH 436 Spring 03 Homework 6 – Solutions

1) As the slope of the graph $f(x)=x^2$ increases unbounded in $(-\infty,+\infty)$ we deduce that f is not Lipschitzian in $(-\infty,+\infty)$. To prove our claim we have to show that

$$\forall_{M\in\mathbb{R}} \quad \exists_{x,y\in(-\infty,+\infty)} \quad |x^2-y^2| > M|x-y|.$$

Let M be given. Take $x=2M$, $y=M$. We check easily that, for such x,y, $|x^2-y^2|>M|x-y|$.

2) f is continuous in $[a,b]$ and differentiable in (a,b). $f'(x)=\cos(x)$ for $x\in(a,b)$. Thus f' is bounded in (a,b). By a theorem proved in class, f is Lipschitzian in $[a,b]$.

3) Since f is Lipschitzian in (a,b), there exists $M\in\mathbb{R}$ such that

$$\forall_{x\in(a,b)} \quad \forall_{y\in(a,b)} \quad |f(x)-f(y)|\le M|x-y|. \quad (1)$$

We have to prove that

$$\forall_{x\in[a,b]} \quad \forall_{y\in[a,b]} \quad |f(x)-f(y)|\le M|x-y|. \quad (2)$$

Fix $x\in(a,b)$. For every $n=2,1,\dots$ take $y_n=a+\frac{b-a}{n}$. Then $y_n\in(a,b)$ and (1) gives that

$$|f(x)-f(y_n)|\le M|x-y_n|. \quad (3)$$

As f is continuous in $[a,b]$, $y_n\to a$ as $n\to+\infty$, (3) implies

$$|f(x) - f(a)| \le M|x-a|. \quad (4)$$

Take $y_n = b - \frac{b-a}{n}$ for $n = 2, 3, \ldots$. Since $y_n \in (a,b)$, (1) gives

$$|f(x) - f(y_n)| \le M|x - y_n|.$$

Since $y_n \to b$ as $n \to +\infty$, we conclude

$$|f(x) - f(b)| \le M|x-b|. \quad (5)$$

As $x \in (a,b)$ was arbitrary, (4) and (5) imply that

$$\forall_{x \in (a,b)} \ \forall_{y \in [a,b]} \quad |f(x) - f(y)| \le M|x-y|. \quad (6)$$

By taking $x_n = a + \frac{b-a}{n}$ and then $x_n = b - \frac{b-a}{n}$ for $n = 2, 3, \ldots$ with any fixed $y \in [a,b]$, and using (6) we prove (2).