MTH 436 Spring 2003 - Homework 3 - Solutions

1) Proof Th 32.4 (c): Assume that $f'(c) < 0$ for all interior points of I. To show that f is strictly decreasing in I, take $x, y \in I$, $y > x$. f satisfies the assumptions of the MVT in $[x, y]$. Hence, there exists $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}$$

Since $f'(c) < 0$, $y - x > 0$, we have $f(y) - f(x) < 0$. Thus $f(y) < f(x)$. Since $x, y \in I$ were arbitrary, we have proved that f is strictly decreasing.

(d) can be proved similarly.

3) Let $h = f - g$. Then h is continuous on I and for each interior point x of I we have

$$h'(x) = f'(x) - g'(x) = 0.$$

Thus, by Th. 32.3, h is constant in I; that is, for some constant c:

$$h(x) = f(x) - g(x) = c \quad \text{for all } x \in I.$$

The latter implies

$$f(x) = g(x) + c \quad \text{for all } x \in I.$$

2) The converse of (a) is not true. Take, for example, $f(x) = x^3$. f is strictly increasing in $I = (-\infty, +\infty)$, yet $f'(0) = 0$.

The converse of (b) is:

"If f is increasing in I, then $f'(x) \geq 0$ for all interior points x of I."

The latter implication is true. Assume f is increasing in I. Let x be an interior point of I. Since f is differentiable at x we have

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} = \lim_{y \to x^+} \frac{f(y) - f(x)}{y - x}.$$

Since f is increasing

$$\frac{f(y) - f(x)}{y - x} \geq 0 \quad \text{whenever } y \in I,\ y > x.$$

Thus

$$f'(x) = \lim_{y \to x^+} \frac{f(y) - f(x)}{y - x} \geq 0.$$

4) For the sake of contradiction suppose that f has two fixed points $d_1, d_2 \in \mathbb{R}$. Without any loss of generality, we may assume $d_1 < d_2$. Notice that f satisfies the assumptions of the MVT in $[d_1, d_2]$

Thus, there exists $c \in (d_1, d_2)$ such that

$$f'(c) = \frac{f(d_2) - f(d_1)}{d_2 - d_1}.$$

Since d_1, d_2 are fixed points $f(d_1) = d_1$, $f(d_2) = d_2$.

Therefore

$$f'(c) = \frac{d_2 - d_1}{d_2 - d_1} = 1.$$

Contradiction as $f'(c) < 1$. We conclude that f has at most one fixed point.

5) (a) Observe that $p(0) = 1$, $p(-1) = -2$. The IVT gives that there exists $c \in (-1, 0)$ such that $p(c) = 0$.

(b) As $p(x) > 0$ for all $x \geq 0$, all roots of $p(x)$ must be negative. Let r_1, r_2 be two distinct negative roots of p, $r_1 < r_2$. From Rolle's Theorem, there exists $d \in (r_1, r_2)$ such that $p'(d) = 0$. (Indeed, $p(r_1) = p(r_2) = 0$.) But $p'(x) = 7x^6 + 5x^4 + 3x^2 > 0$ for all $x < 0$. Contradiction. Therefore, p has exactly one real root.