1) Let f be defined in a nbhd N_f of x_0, g be defined in a nbhd N_g of x_0. Then $f+g$ is defined in $N_f \cap N_g$ which is a nbhd of x_0. Let's examine the limit

$$\lim_{x\to x_0} \frac{(f+g)(x)-(f+g)(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{(f(x)-f(x_0))+(g(x)-g(x_0))}{x-x_0} =$$

$$= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} + \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} = f'(x_0)+g'(x_0).$$

The second equality holds as both limits $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ and $\lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0}$ exist.

2) Let $a \in \mathbb{R}$ be given. We shall prove the following

Lemma: If $a = 0$, then

$$\lim_{x\to a} f(x) = f(a) = 0.$$

If $a \neq 0$, then

$$\lim_{x\to a} f(x)$$

does not exist.

▲

(A "lemma" is a proposition helpful to prove a bigger theorem.)

<u>Proof of the lemma</u>: Let $a=0$. Let a sequence $\{x_n\}_{n=1}^{\infty}$ such that

$$x_n \to 0 \quad \text{as} \quad n \to +\infty \qquad (1)$$

be given. We shall show that

$$f(x_n) \to 0 \quad \text{as} \quad n \to +\infty. \quad (2)$$

Take $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that

$$|x_n| < \varepsilon \quad \text{for } n \geq N. \quad (3)$$

(Such N exists from (1).) The definition of f and (3) imply that

$$|f(x_n)| < \varepsilon \quad \text{for } n \geq N.$$

Hence, (2). By the sequential definition of the limit we conclude that

$$\lim_{x \to 0} f(x) = 0 = f(0).$$

To prove the second part of the lemma take $a \neq 0$. Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ be sequences such that $x_n \in \mathbb{Q}$, $y_n \in \mathbb{R} \setminus \mathbb{Q}$ for $n \in \mathbb{N}$, and $x_n \to a$, $y_n \to a$ as $n \to +\infty$.

Since $f(x_n) = x_n$, $f(y_n) = 0$ for $n \in \mathbb{N}$ we have:

$$f(x_n) \to a, \quad f(y_n) \to 0 \quad \text{as } n \to +\infty.$$

By the sequential definition of the limit we conclude that the limit

$$\lim_{x \to a} f(x)$$

does not exist.

The lemma implies that f is not continuous at any point $a \neq 0$. Thus, f is not differentiable at any point $a \neq 0$.

Let's examine the existence of $f'(0)$. For any $x \neq 0$ we have

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = \begin{cases} 1 & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ irrational.} \end{cases}$$

Thus the limit

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

does not exist. We conclude that f is not differentiable at $a = 0$ either. (Although, f is continuous at $a = 0$.)

4) The product rule cannot be applied as $h(x) = |x|$ is not differentiable at 0. The rest follows from #5 below: $g'(0)$ exists and $g'(0) = |0| = 0$.

5) Since g is continuous at 0, g is defined in a nbhd of 0. Thus f is defined in a nbhd of 0. We have

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x\,g(x) - 0g(0)}{x} = \lim_{x \to 0} g(x) = g(0).$$

Thus $f'(0)$ exists, and $f'(0) = g(0)$.