MTH 436 Spring 03 Homework 1 – Solutions

1) We have to show that

$$\exists_{\varepsilon_0>0}\ \forall_{\delta>0}\ \exists_{x_\delta, y_\delta \in [0,+\infty)} \left(|x_\delta - y_\delta| < \delta \wedge |x_\delta^2 - y_\delta^2| \geq \varepsilon_0 \right). \quad (1)$$

Let $\varepsilon_0 = 1$. Take any $\delta > 0$. Let $n_\delta \in \mathbb{N}$ be such that

$$n_\delta > \frac{1}{\delta}. \quad (2)$$

Take

$$x_\delta = n_\delta, \quad y_\delta = n_\delta + \frac{\delta}{2}.$$

Then

$$|x_\delta - y_\delta| = \frac{\delta}{2} < \delta.$$

But

$$|y_\delta^2 - x_\delta^2| = 2 \cdot n_\delta \cdot \frac{\delta}{2} + \frac{\delta^2}{4} > n_\delta \delta > 1.$$

The latter inequality follows from (2). Thus (1) holds with $\varepsilon_0 = 1$.

2) As we proved in class $g(x)$ is continuous in $[-1,1]$. Since the interval is closed and bounded, $g(x)$ is uniformly continuous in $[-1,1]$.

3) Let $f(x) = mx + b$, where m, b are given constants. We have to show that

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x,y\in(-\infty,+\infty)} \left(|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \right). \quad (3)$$

Observe that for any x, y:

$$|f(x) - f(y)| = |mx - my| = |m||x-y|.$$

For any given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{|m|+1}$. Then

$$|x-y| < \delta \text{ implies } |f(x)-f(y)| < |m| \cdot \frac{\varepsilon}{|m|+1} \leq \varepsilon.$$

Hence, (3) holds.

4) Since f, g are bounded on I, there exist constants M, K such that $M > 0$, $K > 0$ and

$$|f(x)| < M, \quad |g(x)| < K \quad \text{for all } x \in I. \quad (4)$$

To prove that $f \cdot g$ is uniformly continuous in I, take $\varepsilon > 0$. Since f and g are both uniformly continuous in I and $\frac{\varepsilon}{2M} > 0$, $\frac{\varepsilon}{2K} > 0$, there exist $\delta_1 > 0$, $\delta_2 > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2K} \quad \text{whenever } |x-y| < \delta_1, \ x, y \in I, \quad (5)$$

and

$$|g(x) - g(y)| < \frac{\varepsilon}{2M} \quad \text{whenever } |x-y| < \delta_2, \ x, y \in I. \quad (6)$$

Take $\delta = \min\{\delta_1, \delta_2\}$. Then for any $x, y \in I$ such that $|x-y| < \delta$ both (5) and (6) hold. Thus:

$$|f(x)g(x) - f(y)g(y)| = |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \leq$$

$$\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| <$$

$$< M|g(x) - g(y)| + K|f(x) - f(y)| < \varepsilon.$$

The latter two inequalities follow from (4), (5), and (6).

Therefore

$$|f(x)g(x) - f(y)g(y)| < \varepsilon \quad \text{whenever } |x-y| < \delta, \ x, y \in I.$$

Since ε was arbitrary, we conclude that $f \cdot g$ is uniformly continuous in I.