Class Worksheet 2, 4/28/22 - Solutions

Example 1:

Use the Fundamental Theorem of Line Integrals to calculate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = x^3 \vec{i} + e^{10y} \vec{j}$ and C is the unit circle oriented clockwise.

Enter an exact answer.

$\int_C \vec{F} \cdot d\vec{r} =$

Solution

Since

$$\vec{F} = x^3 \vec{i} + e^{10y} \vec{j} = \nabla \left(\frac{1}{4} x^4 + \frac{1}{10} e^{10y} \right),$$

we see $\vec{F}$ is a gradient vector field. Therefore, since C is a closed cycle,

$$\int_C \left(x^3 \vec{i} + e^{10y} \vec{j} \right) \cdot d\vec{r} = 0.$$

Example 2:

Use the Fundamental Theorem of Line Integrals to calculate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = y\sin(xy)\,\vec{i} + x\sin(xy)\,\vec{j}$ and C is the parabola $y = 2x^2$ from $(2, 8)$ to $(9, 162)$.

NOTE: Enter the exact answer, or round to three decimal places.

Solution:

Since $\vec{F} = y\cos(xy)\,\vec{i} + x\sin(xy)\,\vec{j} = \text{grad}\,(-\cos(xy))$, the Fundamental Theorem of Line Integrals gives:

$$\int_C \vec{F} \cdot d\vec{r} = -\cos xy\Big|_{(2,8)}^{(9,162)} = \cos(16) - \cos(1458)$$

Example 3:

Use the Fundamental Theorem of Line Integrals to calculate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = 4\sin(4x+y)\vec{i} + \sin(4x+y)\vec{j}$ and C is the path consisting of a line from $(6\pi, 0)$ to $(2, 5)$ followed by a line to $(7\pi, 0)$ followed by a quarter circle to $(0, 7\pi)$.

Enter an exact answer.

$\int_C \vec{F} \cdot d\vec{r} =$

Solution

Since $\vec{F} = 4\sin(4x+y)\vec{i} + \sin(4x+y)\vec{j} = \nabla(-\cos(4x+y))$, we take

$f(x, y) = -\cos(4x+y)$.

Then, using the Fundamental Theorem of Line Integrals,

$$
\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= f(0, 7\pi) - f(6\pi, 0) \\
&= -\cos(7\pi) - (-\cos(24\pi)) \\
&= -(-1) - (-1) \\
&= 2.
\end{aligned}
$$

Example 4:

If $\vec{F} = \text{grad}(x^3 + y^5)$, find $\int_C \vec{F} \cdot d\vec{r}$ where C is the quarter of the circle $x^2 + y^2 = 16$ in the first quadrant, oriented counterclockwise.

Enter an exact answer.

$$\int_C \vec{F} \cdot d\vec{r} =$$

Solution

Since $\vec{F}$ is a gradient field, with $\vec{F} = \text{grad} f$ where $f(x, y) = x^3 + y^5$, we use the Fundamental Theorem of Line Integrals. The starting point of the path C is $(4, 0)$ and the end is $(0, 4)$. Thus,

$$\int_C \vec{F} \cdot d\vec{r} = f(0, 4) - f(4, 0) = 1024 - 64 = 960.$$