Class Worksheet 1, 4/28/22 - Solutions

For each of the problems below, find a given line integral using a parametric representation of the path. Check if it is possible to find the integral using the Fundamental Theorem for Line Integrals (FTLI). If yes, find the integral using FTLI as well and compare your answers.

Example 1:

Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = y^3\,\vec{i} + x^2\,\vec{j}$ and C is the line from $(0, 0)$ to $(12, 4)$.

Enter an exact answer.

$\int_C \vec{F} \cdot d\vec{r} =$

Solution

Parameterizing C by $x(t) = 12t,\ y(t) = 4t$ for $0 \le t \le 1$, we have $d\vec{r} = \left(12\,\vec{i} + 4\,\vec{j}\right)dt$, so

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \left((4t)^3\,\vec{i} + (12t)^2\,\vec{j}\right) \cdot \left(12\,\vec{i} + 4\,\vec{j}\right) dt \\
&= \int_0^1 \left(768t^3 + 576t^2\right) dt \\
&= \left(192t^4 + 192t^3\right)\Big|_0^1 \\
&= 384\ .
\end{aligned}$$

The integral cannot be calculated using FTLI as

$$\frac{\partial F_1}{\partial y} = 3y^2 \neq \frac{\partial F_2}{\partial x} = 2x.$$

Hence, the vector field does not have a potential function.

Example 2:

Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = x^2\vec{i} + y^2\vec{j}$ and C is the line from $(3, 4)$ to $(6, 7)$.

Enter an exact answer.

$\int_C \vec{F} \cdot d\vec{r} =$

Solution

The line can be parameterized by $(3 + 3t, 4 + 3t)$, for $0 \le t \le 1$, so the integral looks like

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{F}\,(3 + 3t, 4 + 3t) \cdot \left(3\vec{i} + 3\vec{j}\right)dt \\
&= \int_0^1 \left[(3 + 3t)^2\,\vec{i} + (4 + 3t)^2\,\vec{j}\right] \cdot \left(3\vec{i} + 3\vec{j}\right)dt \\
&= \int_0^1 \left[3\left(9 + 18t + 9t^2\right) + 3\left(16 + 24t + 9t^2\right)\right]dt \\
&= \int_0^1 \left(75 + 126t + 54t^2\right)dt \\
&= 75 + 63 + \tfrac{54}{3} = 156\,.
\end{aligned}$$

The vector field $\vec{F}$ has a potential function $f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3$. Indeed, $\vec{F} = \nabla f$. By FTLI we have:

$$\int_C \vec{F} \cdot d\vec{r} = f(6,7) - f(3,4) = 156.$$

Example 3:

Find the line integral $\int_C \left(3y^2 \vec{i} + x \vec{j}\right) \cdot d\vec{r}$ where C is the line segment from $(5, 1)$ to $(0, 0)$.

Enter an exact answer.

$$\int_C \left(3y^2 \vec{i} + x \vec{j}\right) \cdot d\vec{r} =$$

Solution

We will find the line integral from $(0, 0)$ to $(5, 1)$ and then take the negative. The line segment is parameterized by

$x = 5t, \quad y = t,$ for $0 \leq t \leq 1.$

Then $d\vec{r} = \left(5\vec{i} + \vec{j}\right)dt$, so

$$\begin{aligned}
\int_C \left(3y^2 \vec{i} + x \vec{j}\right) \cdot d\vec{r} &= -\int_0^1 \left(3t^2 \vec{i} + 5t\vec{j}\right) \cdot \left(5\vec{i} + \vec{j}\right) dt \\
&= -\int_0^1 \left(15t^2 + 5t\right) dt \\
&= -\left(5t^3 + \frac{5}{2}t^2\right)\Big|_0^1 \\
&= -\frac{15}{2}.
\end{aligned}$$

The integral cannot be calculated using FTLI as

$$\frac{\partial F_1}{\partial y} = 6y \neq \frac{\partial F_2}{\partial x} = 1.$$

Hence, the vector field does not have a potential function.

Example 4:

Find the line integral $\int_C \left(x\vec{i} + y\vec{j} \right) \cdot d\vec{r}$ where C is the semicircle with center at $(2, 0)$ and going from $(4, 0)$ to $(0, 0)$ in the region $y > 0$.

Enter an exact answer.

$\int_C \left(x\vec{i} + y\vec{j} \right) \cdot d\vec{r}$ =

Solution

The semicircle has radius 2 and is centered at $(2, 0)$. It can be parameterized by

$x = 2 + 2\cos t, \quad y = 2\sin t$, for $0 \le t \le \pi$.

Then $\vec{r}\,'(t) = -2\sin t\,\vec{i} + 2\cos t\,\vec{j}$, so

$$\begin{aligned}
\int_C \left(x\vec{i} + y\vec{j} \right) \cdot d\vec{r} &= \int_0^\pi \left((2 + 2\cos t)\vec{i} + 2\sin t\,\vec{j} \right) \cdot \left(-2\sin t\,\vec{i} + 2\cos t\,\vec{j} \right) dt \\
&= \int_0^\pi (-4\sin t - 4\cos t\sin t + 4\sin t\cos t)\, dt \\
&= -4\cos t\Big|_0^\pi \\
&= -8.
\end{aligned}$$

The vector field $\vec{F}$ has a potential function $f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Indeed, $\vec{F} = \nabla f$. By FTLI we have:

$$\int_C \vec{F} \cdot d\vec{r} = f(0, 0) - f(4, 0) = -8.$$