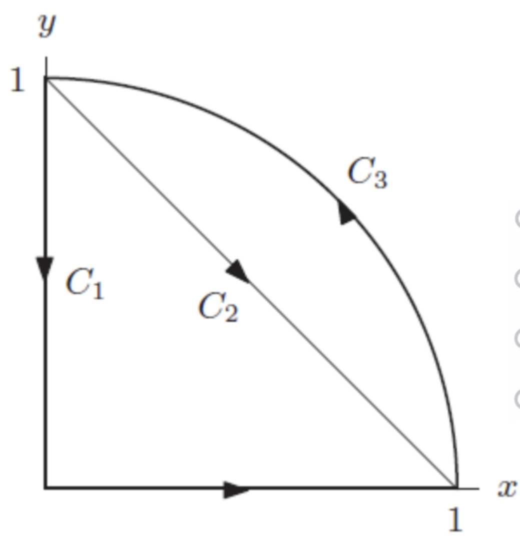
Class Worksheet 4/25/22 - Solutions

Example 1:

Which of the following paths gives the same line integral of $\vec{F}$ = *grad f*?



- (a) C_1 and C_2
- (b) C_2 and C_3
- (c) C_1 and C_3
- (d) All the line integrals are different

Solution

Answer: (a)

Example 2:

Use the Fundamental Theorem of Line Integrals to calculate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = 17x^{16}\vec{i} + 5y^4\vec{j}$ and C is the top of the unit circle from $(1, 0)$ to $(-1, 0)$.

Solution

Since $\vec{F} = 17x^{16}\vec{i} + 5y^4\vec{j} = \nabla\left(x^{17} + y^5\right)$, we take $f(x, y) = x^{17} + y^5$. Then by the Fundamental Theorem of Line Integrals,

$$\int_C \vec{F} \cdot d\vec{r} = f(-1, 0) - f(1, 0) = (-1)^{17} - 1^{17} = -2.$$

Example 3:

Let $\vec{F} = \left(x^2 + 3x^2y^4\right)\vec{i} + 4x^3y^3\vec{j}$. Let C_1 be the path along the x-axis from $(8,0)$ to $(-8,0)$; let C_2 be the semi-circle in the upper half plane from $(8,0)$ to $(-8,0)$.

Find the exact values of the following integrals.

(a) $\int_{C_1} \vec{F} \cdot d\vec{r} =$

(b) $\int_{C_2} \vec{F} \cdot d\vec{r} =$

Solution

(a) Since $\vec{F} = \nabla\left(\frac{x^3}{3} + x^3y^4\right)$, the Fundamental Theorem of Line Integrals gives

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \frac{x^3}{3} + x^3y^4\Big|_{(8,0)}^{(-8,0)} = -\frac{512}{3} + 0 - \left(\frac{512}{3} + 0\right) = -\frac{1024}{3}.$$

(b) Since a gradient field is path-independent, and the endpoints of C_1 and C_2 are the same:

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} = -\frac{1024}{3}.$$

Example 4:

If $\overrightarrow{F}$ is a path-independent vector field, with $\int_{(0,0)}^{(1,0)} \overrightarrow{F} \cdot d\overrightarrow{r} = 5.7$ and $\int_{(1,0)}^{(1,1)} \overrightarrow{F} \cdot d\overrightarrow{r} = 3.5$ and $\int_{(0,1)}^{(1,1)} \overrightarrow{F} \cdot d\overrightarrow{r} = 4.6$, find

$\int_{(0,1)}^{(0,0)} \overrightarrow{F} \cdot d\overrightarrow{r}$.

Solution

Since the vector field is path independent, the line integral around the closed curve $(0, 0)$ to $(1, 0)$ to $(1, 1)$ to $(0, 1)$ to $(0, 0)$ is zero.
We have

$$\begin{aligned}
\int_{(0,1)}^{(0,0)} \overrightarrow{F} \cdot d\overrightarrow{r} &= -\left(\int_{(0,0)}^{(1,0)} \overrightarrow{F} \cdot d\overrightarrow{r} + \int_{(1,0)}^{(1,1)} \overrightarrow{F} \cdot d\overrightarrow{r} + \int_{(1,1)}^{(0,1)} \overrightarrow{F} \cdot d\overrightarrow{r}\right) \\
&= -(5.7 + 3.5 - 4.6) \\
&= -4.6.
\end{aligned}$$

Thus, $\int_{(0,1)}^{(0,0)} \overrightarrow{F} \cdot d\overrightarrow{r}$ equals the integral from $(0, 1)$ to $(1, 1)$ to $(1, 0)$ to $(0, 0)$ which equals $4.6 - 3.5 - 5.7 = -4.6$.