Class Worksheet 4/21/22 - Solutions

Example 1:

Find $\int_C \vec{F} \cdot d\vec{r}$ when $\vec{F} = x\vec{i} + y\vec{j}$ and C is the line from $(0, 0)$ to $(6, 6)$.

Enter an exact answer.

$\int_C \vec{F} \cdot d\vec{r} =$

Solution

The curve C is parameterized by $x = t, y = t$ for $0 \leq t \leq 6$. Thus

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^6 \left(t\vec{i} + t\vec{j}\right) \cdot \left(\vec{i} + \vec{j}\right) dt = \int_0^6 2t\, dt = t^2 \Big|_0^6 = 36.$$

Example 2:

Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = x^2\vec{i} + y^2\vec{j}$ and C is the line from $(5, 4)$ to $(8, 7)$.

Enter an exact answer.

$\int_C \vec{F} \cdot d\vec{r} =$

Solution

The line can be parameterized by $(5 + 3t, 4 + 3t)$, for $0 \leq t \leq 1$, so the integral looks like

$$
\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{F}(5 + 3t, 4 + 3t) \cdot \left(3\vec{i} + 3\vec{j}\right)dt \\
&= \int_0^1 \left[(5 + 3t)^2\vec{i} + (4 + 3t)^2\vec{j}\right] \cdot \left(3\vec{i} + 3\vec{j}\right)dt \\
&= \int_0^1 \left[3\left(25 + 30t + 9t^2\right) + 3\left(16 + 24t + 9t^2\right)\right]dt \\
&= \int_0^1 \left(123 + 162t + 54t^2\right)dt \\
&= 123 + 81 + \tfrac{54}{3} = 222\,.
\end{aligned}
$$

Example 3:

Find the line integral $\int_C \left(x\,\vec{i} + y\,\vec{j}\right) \cdot d\,\vec{r}$ where C is the semicircle with center at $(2, 0)$ and going from $(6, 0)$ to $(-2, 0)$ in the region $y > 0$.

Enter an exact answer.

$$\int_C \left(x\,\vec{i} + y\,\vec{j}\right) \cdot d\,\vec{r} =$$

Solution

The semicircle has radius 4 and is centered at $(2, 0)$. It can be parameterized by

$x = 2 + 4\cos t, \quad y = 4\sin t$, for $0 \le t \le \pi$.

Then $\vec{r}\,'(t) = -4\sin t\,\vec{i} + 4\cos t\,\vec{j}$, so

$$\begin{aligned}\int_C \left(x\,\vec{i} + y\,\vec{j}\right) \cdot d\,\vec{r} &= \int_0^{\pi} \left((2 + 4\cos t)\,\vec{i} + 4\sin t\,\vec{j}\right) \cdot \left(-4\sin t\,\vec{i} + 4\cos t\,\vec{j}\right) dt \\ &= \int_0^{\pi} (-8\sin t - 16\cos t\sin t + 16\sin t\cos t)\,dt \\ &= 8\cos t\Big|_0^{\pi} \\ &= -16.\end{aligned}$$

Example 4:

Evaluate the line integral of the vector field

$\vec{F} = (3x - y)\vec{i} + x\vec{j}$

along the path (t^2, t) with $0 \le t \le 1$.

Enter an exact answer.

$\int_C \vec{F} \cdot d\vec{r} =$

<u>Solution</u>

Since $\vec{r}(t) = t^2\vec{i} + t\,\vec{j}$, we have $d\,\vec{r} = \left(2\text{t}\,\vec{i} + \vec{j}\right)dt$. Thus

$$\begin{aligned}
\int_C \vec{F} \cdot d\,\vec{r} &= \int_0^1 \vec{F}\,(t^2, t) \cdot \left(2\text{t}\,\vec{i} + \vec{j}\right)dt \\
&= \int_0^1 \left[(3t^2 - t)\,\vec{i} + t^2\,\vec{j}\,\right] \cdot \left(2\text{t}\,\vec{i} + \vec{j}\right)dt \\
&= \int_0^1 (6t^3 - t^2)\,dt \\
&= \left(\frac{3t^4}{2} - \frac{t^3}{3}\right)\Big|_0^1 = \frac{3}{2} - \frac{1}{3} - (0 - 0) = \frac{7}{6}.
\end{aligned}$$