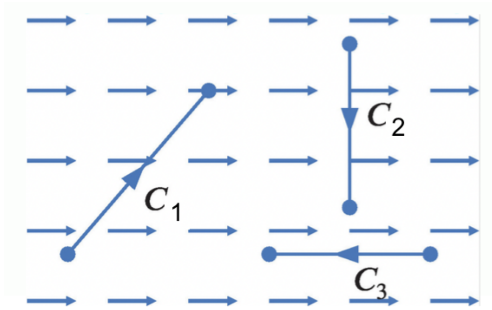
Class Worksheet 4/19/22 - Solutions

Example 1:

Consider the vector field $\vec{F}$ shown in the figure below, together with the paths C_1 , C_2 , and C_3 .



Arrange the line integrals $\int_{C_1} \vec{F} \cdot d\vec{r}$, $\int_{C_2} \vec{F} \cdot d\vec{r}$ and $\int_{C_3} \vec{F} \cdot d\vec{r}$ in ascending order.

- ○ $\int_{C_2} \vec{F} \cdot d\vec{r} < \int_{C_3} \vec{F} \cdot d\vec{r} < \int_{C_1} \vec{F} \cdot d\vec{r}$
- ○ $\int_{C_3} \vec{F} \cdot d\vec{r} < \int_{C_2} \vec{F} \cdot d\vec{r} < \int_{C_1} \vec{F} \cdot d\vec{r}$
- ○ $\int_{C_2} \vec{F} \cdot d\vec{r} < \int_{C_1} \vec{F} \cdot d\vec{r} < \int_{C_3} \vec{F} \cdot d\vec{r}$
- ○ $\int_{C_1} \vec{F} \cdot d\vec{r} < \int_{C_3} \vec{F} \cdot d\vec{r} < \int_{C_2} \vec{F} \cdot d\vec{r}$
- ○ $\int_{C_3} \vec{F} \cdot d\vec{r} < \int_{C_1} \vec{F} \cdot d\vec{r} < \int_{C_2} \vec{F} \cdot d\vec{r}$

Solution

Since it appears that C_2 is everywhere perpendicular to the vector field, all of the dot products in the line integral are zero, hence

$$\int_{C_2} \vec{F} \cdot d\vec{r} = 0.$$

Along the path C_1 the dot products of $\vec{F}$ with $\Delta \vec{r_i}$ are all positive, so their sum is positive and we have

$$\int_{C_2} \vec{F} \cdot d\vec{r} < \int_{C_1} \vec{F} \cdot d\vec{r}.$$

For C_3 the vectors $\Delta \vec{r_i}$ are in the opposite direction to the vectors of $\vec{F}$, so the dot products $\vec{F} \cdot \Delta \vec{r}_i$ are all negative; so,

$$\int_{C_3} \vec{F} \cdot d\vec{r} < 0.$$

Thus, we have

$$\int_{C_3} \vec{F} \cdot d\vec{r} < \int_{C_2} \vec{F} \cdot d\vec{r} < \int_{C_1} \vec{F} \cdot d\vec{r}.$$

Example 2: Calculate the line integral of the vector field $\vec{F}(x, y) = 3\vec{i} + 4\vec{j}$ along the segment from $(0, 6)$ to $(0, 13)$.

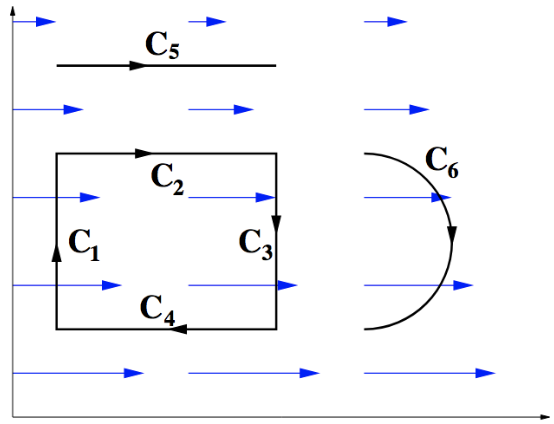
Solution:

Since $\vec{F}$ is a constant vector field and the curve is a line, $\int_C \vec{F} \cdot d\vec{r} = \vec{F} \cdot \Delta\vec{r}$, where $\Delta\vec{r} = 7\vec{j}$. Therefore,

$$\int_C \vec{F} \cdot d\vec{r} = (3\vec{i} + 4\vec{j}) \cdot 7\vec{j} = 28$$

Example 3:

Given to the right is a vector field $\vec{F}(x,y)$, together with oriented curves C_1, C_2, C_3, C_4, C_5, and C_6.



(a) Is $\int_{C_6} \vec{F} \cdot d\vec{r}$ positive, negative, or zero? Explain.

Negative. The path is semicircular, and we travel for equal distances along C with and against the flow of the vector field. However, the vector field has a greater magnitude along the portion of the path in which we are traveling against the flow of the vector field.

(b) Let C be the closed curve $C_1 + C_2 + C_3 + C_4$. Is $\int_C \vec{F} \cdot d\vec{r}$ positive, negative, or zero? Explain.

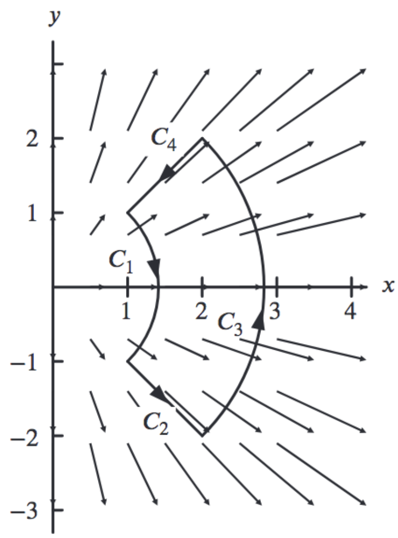
First, we note that C_1 and C_3 are perpendicular to the flow of the vector field, so we have

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \underbrace{\int_{C_1} \vec{F} \cdot d\vec{r}}_{\overset{\downarrow}{0}} + \int_{C_2} \vec{F} \cdot d\vec{r} + \underbrace{\int_{C_3} \vec{F} \cdot d\vec{r}}_{\overset{\downarrow}{0}} + \int_{C_4} \vec{F} \cdot d\vec{r} \\ &= \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_4} \vec{F} \cdot d\vec{r}\end{aligned}$$

Now, $\int_{C_2} \vec{F} \cdot d\vec{r} > 0$ because we travel with the vector field along C_2, and $\int_{C_4} \vec{F} \cdot d\vec{r} < 0$ because we travel against the vector field along C_4. However, the magnitude of the vector field is greater along C_4, which means that

$$\int_C \vec{F} \cdot d\vec{r} \quad \text{is} \quad \underline{\text{negative}}.$$

Example 4: The vector field depicted below is $\vec{F}(x, y) = x\vec{i} + y\vec{j}$. Let $C = C_1 + C_2 + C_3 + C_4$. Is $\int_C \vec{F} \cdot d\vec{r}$ positive negative or zero? Explain.



The vector field is $F(\vec{r}) = \vec{r}$.

The vector field is perpendicular to the circular arcs at every point, so

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_3} \vec{F} \cdot d\vec{r} = 0.$$

Also, since it is radially symmetric,

$$\int_{C_2} \vec{F} \cdot d\vec{r} = -\int_{C_4} \vec{F} \cdot d\vec{r}.$$

So,

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0.$$