Class Worksheet 3/8/2022

Example 1:

Given that $z = \sin\left(\frac{x}{y}\right)$, $x = \ln u$ and $y = v$, find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

Assume the variables are restricted to the domain on which the functions are defined.

NOTE: Enter exact answers as a functions of u and v only.

Solution:

Since z is a function of two variables, x and y, which are functions of two variables, u and v, the two chain rule identites which apply are:

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\,\frac{\partial y}{\partial u} \\
&= \left(\cos\left(\frac{x}{y}\right)\right)\left(\frac{1}{y}\right)\frac{1}{u} + \left(\cos\left(\frac{x}{y}\right)\right)\left(\frac{-x}{y^2}\right)\cdot 0 \\
&= \frac{1}{vu}\cos\left(\frac{\ln u}{v}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\,\frac{\partial y}{\partial v} \\
&= \left(\cos\left(\frac{x}{y}\right)\right)\left(\frac{1}{y}\right)\cdot 0 + \left(\cos\left(\frac{x}{y}\right)\right)\left(\frac{-x}{y^2}\right)\cdot 1 \\
&= -\frac{\ln u}{v^2}\cos\left(\frac{\ln u}{v}\right)
\end{aligned}$$

Example 2:

Let $z = f(x, y)$ where $x = g(t)$, $y = h(t)$ and f, g, h are all differentiable functions.

Given the information in the table, find $\left.\dfrac{\partial z}{\partial t}\right|_{t=1}$.

$f(3, 10) = 7$	$f(4, 11) = -20$	$g(1) = 3$
$f_x(3, 10) = 250$	$f_y(3, 10) = 0.1$	$g'(1) = 4$
$f_x(4, 11) = 200$	$f_y(4, 11) = 0.2$	$h(1) = 10$
$f(3, 4) = -10$	$f(10, 11) = -1$	$h'(1) = 12$

NOTE: Enter the exact answer.

Solution:
When $t = 1$,

$$x = g(1) = 3 \text{ and } y = h(1) = 10,$$

so

$$z = f(3, 10) = 7.$$

$f(3, 10) = 7$	$f(4, 11) = -20$	$g(1) = 3$
$f_x(3, 10) = 250$	$f_y(3, 10) = 0.1$	$g'(1) = 4$
$f_x(4, 11) = 200$	$f_y(4, 11) = 0.2$	$h(1) = 10$
$f(3, 4) = -10$	$f(10, 11) = -1$	$h'(1) = 12$

Thus

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} g'(t) + \frac{\partial f}{\partial y} h'(t)$$

gives

$$\begin{aligned}\left.\frac{\partial z}{\partial t}\right|_{t=1} &= f_x(3, 10) \cdot g'(1) + f_y(3, 10) \cdot h'(1) \\ &= 250 \cdot 4 + 0.1 \cdot 12 \\ &= 1000 + 1.2 \\ &= 1001.2\end{aligned}$$