Class Worksheet 3/3/2022 - Solutions

Example 1: The concentration of salt in a fluid at (x, y, z) is given by $F(x, y, z) = x^5 + y^{11} + x^5z^2$ mg/cm^3. x, y, z are measured in centimeters. You are at the point $(-1, 1, 1)$. In which direction should you move if you want the concentration to increase the fastest? Give your answer as a unit vector. How fast is the concentration changing in that direction? Give units.

Solution: If you want the concentration to increase the fastest, you should move in the direction of grad$F(-1, 1, 1)$.

$$\begin{aligned} \operatorname{grad} F\,\Big|_{(-1,1,1)} &= \left(\left(5x^4 + 5x^4z^2\right)\vec{i} + 11y^{10}\vec{j} + 2x^5z\,\vec{k}\right)\Big|_{(-1,1,1)} \\ &= 10\,\vec{i} + 11\,\vec{j} - 2\,\vec{k}\,. \end{aligned}$$

The magnitude $||\text{grad}F(-1, 1, 1)|| = 15$. The unit vector in the direction of the gradient is:

$$\vec{u} = \frac{1}{15}(10\vec{i} + 11\vec{j} - 2\vec{k}).$$

The rate of change in the direction of $\vec{u}$ is $||\text{grad}F(-1, 1, 1)|| = 15\dfrac{\frac{\text{mg}}{\text{cm}^3}}{\text{cm}}$.

Example 2:

Find the directional derivative using $f(x, y, z) = xy + z^2$ as you leave the point (4, 4, 0) heading in the direction of the point (0, 4, 1).

NOTE: Enter the exact answer or round to three decimal places.

Solution:

We have grad $f = y\,\vec{i} + x\,\vec{j} + 2z\,\vec{k}$, so

$$\text{grad}\, f(4,\ 4,\ 0) = 4\,\vec{i} + 4\,\vec{j}.$$

We are moving in the direction of (0, 4, 1) − (4, 4, 0) = (−4, 0, 1).

A unit vector in the direction we want is $\vec{u} = \dfrac{1}{\sqrt{17}}\left(-4\,\vec{i} + \vec{k}\right)$. Therefore,

the directional derivative is

$$\text{grad}\, f(4,\ 4,\ 0) \cdot \vec{u} = \frac{4(-4) + 4 \cdot 0 + 0 \cdot 1}{\sqrt{17}} = -\frac{16}{\sqrt{17}}.$$

Example 3:

Check that the point $(-1,\ 1,\ 2)$ lies on the surface $x^2 - y^2 + z^2 = 4$. Then, viewing the surface as a level surface for a function $f(x, y, z)$, find a vector normal to the surface and an equation for the tangent plane to the surface at $(-1,\ 1,\ 2)$.

Solution:
First, we check that $(-1)^2 - (1)^2 + 2^2 = 4$. Then, let $f(x, y, z) = x^2 - y^2 + z^2$ so that the given surface is the level surface $f(x, y, z) = 4$. Since $f_x = 2x$, $f_y = -2y$, and $f_z = 2z$, we have

$$\operatorname{grad} f(-1,\ 1,\ 2) = -2\vec{i} - 2\vec{j} + 4\vec{k}.$$

Since gradients are perpendicular to level surfaces, a vector normal to the surface at $(-1,\ 1,\ 2)$ is $\vec{n} = -2\vec{i} - 2\vec{j} + 4\vec{k}$. Thus, an equation for the tangent plane is

$$-2(x + 1) - 2(y - 1) + 4(z - 2) = 0,$$

or

$$z = \frac{x}{2} + \frac{y}{2} + 2.$$