Solutions 3/1/2022

Class Worksheet

Example 1: Let $f(x, y) = x^2y + 7xy^3$. Find the gradient of f at the point $(1, 2)$.

$f_x = \frac{\partial}{\partial x}[x^2y + 7xy^3] = 2xy + 7y^3$, $f_y = \frac{\partial}{\partial y}[x^2y + 7xy^3] = x^2 + 21xy^2$

$\nabla f(x,y) = (2xy + 7y^3)\vec{i} + (x^2 + 21xy^2)\vec{j}$

At $(x,y) = (1,2)$:

$\nabla f(1,2) = (4+56)\vec{i} + (1+84)\vec{j} = \boxed{60\vec{i} + 85\vec{j} = \nabla f(1,2)}$

Example 2: Find the directional derivative of the function $g(x, y) = x^2y$ at the point $(2, 6)$ in the direction $\vec{v} = 4\vec{i} - 3\vec{j}$.

$\vec{v}$ is not a unit vector. So $g_{\vec{v}}(2,6) = g_{\frac{\vec{v}}{\|\vec{v}\|}}(2,6)$ by definition.

Calculate $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$. $\|\vec{v}\| = \sqrt{16+9} = 5$. So $\vec{u} = \frac{4}{5}\vec{i} - \frac{3}{5}\vec{j}$.

$g_{\vec{u}}(2,6) = \nabla g(2,6) \cdot \vec{u}$.

$g_x = 2xy$, $g_y = x^2$. $\nabla g(2,6) = g_x(2,6)\vec{i} + g_y(2,6)\vec{j} = 24\vec{i} + 4\vec{j}$.

$g_{\vec{u}}(2,6) = \nabla g(2,6) \cdot \vec{u} = \frac{96}{5} - \frac{12}{5} = \frac{84}{5}$

Example 3: The temperature at the point (x, y) of a metal plate is given by $T(x, y) = 1 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ degrees Celsius, where x and y are both measured in centimeters. Find the direction of the fastest increase in temperature at the point $(1, 2)$ of the plate. What is the rate of increase in T if you move from $(1, 2)$ in that direction? Give units with your answer.

The direction of the fastest increase in T at $(1,2)$ is given by $\nabla T(1,2)$. The rate of increase in the direction $\nabla T(1,2)$ is $\|\nabla T(1,2)\|$. We calculate:

$T_x = -\frac{1}{2}x$, $T_y = -\frac{1}{2}y$, $\nabla T(x,y) = (-\frac{1}{2}x)\vec{i} + (-\frac{1}{2}y)\vec{j}$

$\nabla T(1,2) = -\frac{1}{2}\vec{i} - \vec{j}$ – the direction of fastest increase

$\|\nabla T(1,2)\| = \sqrt{\frac{1}{4} + 1} = \sqrt{1.25} \cong 1.12\ \frac{°C}{cm}$ – the rate of increase.