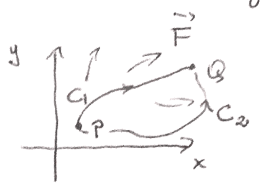
18.3 Path-Independent Vector Fields - Part 2

Recall that a vector field

$$\vec{F}(x,y) = F_1(x,y)\vec{i} + F_2(x,y)\vec{j}$$

is called path-independent (or conservative) if for any two poi

P, Q the integral of $\vec{F}$ along any path from P to Q is the sa



$$\int_{C_1} \vec{F} = \int_{C_2} \vec{F} = \int_P^Q \vec{F}.$$

Let's summarize what we know already about conservative fields.

① $\vec{F}(x,y)$ is conservative if and only if

$$\vec{F}(x,y) = \nabla f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$$

for some function f. f is called a potential function of $\vec{F}$.

If $\vec{F}$ has a potential function then:

$$\int_P^Q \vec{F} = f(Q) - f(P).$$

② $\vec{F}$ is conservative if and only if for any closed path C

$$\int_C \vec{F} = 0.$$

So if we can find a closed path C for which $\int_C \vec{F} \neq 0$, $\vec{F}$ is not conservative.

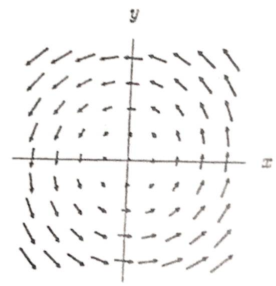
③ If $\vec{F}(x,y) = F_1(x,y)\vec{i} + F_2(x,y)\vec{j}$ is conservative, then

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}.$$

Hence, if the equality doesn't hold $\vec{F}$ is not conservative.

Ex: Is the following vector field conservative?

$\vec{F}$:



No. For every circle C centered at the origin, oriented counterclockwise

$$\int_C \vec{F} > 0$$

and not 0.

Ex: Is $\vec{F}(x,y) = 3x^2y\,\vec{i} - xy\,\vec{j}$ conservative?

No. $F_1(x,y) = 3x^2y$, $F_2(x,y) = -xy$. Hence, $\frac{\partial F_1}{\partial y} = 3x^2 \neq \frac{\partial F_2}{\partial x} = -y$.

If we have a vector field

$$\vec{F}(x,y) = F_1(x,y)\vec{i} + F_2(x,y)\vec{j}$$

and we suspect that $\vec{F}$ might be conservative, how do we go about find a potential function $f(x,y)$? That is, a function such that

$$\nabla f = f_x\vec{i} + f_y\vec{j} = F_1\vec{i} + F_2\vec{j}.$$

Ex: Let $\vec{F}(x,y) = (6xy)\vec{i} + (3x^2 + 12y^2)\vec{j}$. Find a potential function f of $\vec{F}$ if possible.

Apply our simple test: $\frac{\partial F_1}{\partial y} = 6x$, $\frac{\partial F_2}{\partial x} = 6x$. They are equal so a potential function may exist.

A potential function of $\vec{F}$ has to satisfy:

$$f_x(x,y) = 6xy, \quad f_y(x,y) = 3x^2 + 12y^2.$$

Hence:

$$f = \int(6xy)\,dx = 3x^2y + C(y)$$

How to choose $C(y)$ so :

$$f_y = \frac{\partial}{\partial y}[3x^2y + C(y)] = 3x^2 + 12y^2$$

We have to have $C'(y) = 12y^2$. So $C(y) = 4y^3 + K$

We can take :

$$f(x,y) = 3x^2y + 4y^3 + K$$

for any K as our potential function. To use the Fundamental Theorem, we would take $K = 0$, of course.

Ex : Let $\vec{F} = 6\sin(6x+y)\vec{i} + \sin(6x+y)\vec{j}$. Find a potential function f if possible.

We are looking for a function $f(x,y)$ such that:

$$f_x = 6\sin(6x+y) \quad , \quad f_y = \sin(6x+y).$$

So

$$f(x,y) = \int 6\sin(6x+y)dx = -\cos(6x+y) + C(y)$$

$C(y)$ must be such that

$$f_y = \frac{\partial}{\partial y}[-\cos(6x+y) + C(y)] = \sin(6x+y).$$

Since

$$\frac{\partial}{\partial y}[-\cos(6x+y)] = \sin(6x+y)$$

we want $C'(y)$ to be 0. Hence, any function

$$f(x,y) = -\cos(6x+y) + K$$

is a potential of $\vec{F}$.