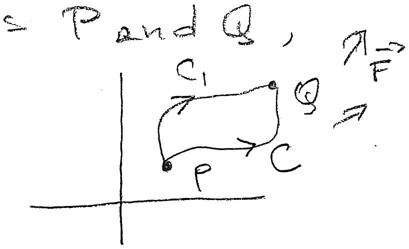
1) No quiz R
2) Final

Apr 21, 23

18.3 Path-independent Vector Fields

<u>Def</u>: A vector field $\vec{F}$ is said to be path-independent (or conservative), if for any two points P and Q, the line integral



$$\int_C \vec{F}\,d\vec{r}$$

is the same for any path C from P to Q. △

If $\vec{F}$ is conservative, we can simply write:

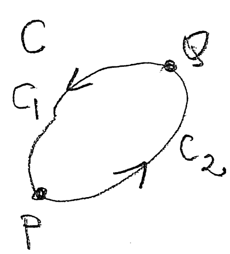
$$\int_P^Q \vec{F}\,d\vec{r}.$$

It seems like a strange property. Are there any p-i fields? We shall soon see. The first important observation:

<u>Th</u>: $\vec{F}$ is conservative if and only if for any closed path C:

$$\oint_C \vec{F}\,d\vec{r} = 0.$$ △

Indeed. "⇒" Let $\vec{F}$ be conservative and C be closed. Choose and two points P and Q on C, take C_1 and C_2 as on the picture. Then:

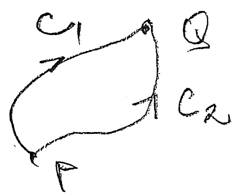


$$\int_P^Q \vec{F}\,d\vec{r} = \int_{C_2} \vec{F}\,d\vec{r} = \int_{-C_1} \vec{F}\,d\vec{r} = -\int_{C_1} \vec{F}\,d\vec{r},$$

So

$$\oint_C \vec{F}\,d\vec{r} = \int_{C_2} \vec{F}\,d\vec{r} - \left(-\int_{C_1} \vec{F}\,d\vec{r}\right) = 0.$$

Similarly "$\Leftarrow$". Suppose $\oint_C \vec{F}\,d\vec{r} = 0$ for any closed path C.
To show that $\vec{F}$ is conservative take any P and Q and any two paths C_1, C_2 from P to Q:



Then:

$$\int_{C_1} \vec{F} + \int_{-C_2} \vec{F} = 0 = \int_{C_1} \vec{F} - \int_{C_2} \vec{F} = 0$$

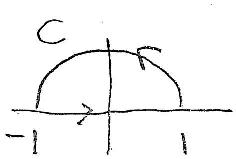
Thus:

$$\int_{C_1} \vec{F}\,d\vec{r} = \int_{C_2} \vec{F}\,d\vec{r}.$$

So

$$\underline{\vec{F} \text{ conservative} \iff \oint_C \vec{F}\,d\vec{r} = 0 \text{ for any closed path.}}$$

<u>Ex</u>: Recall: $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$



$$\oint_C \vec{F}\,d\vec{r} = \pi.$$

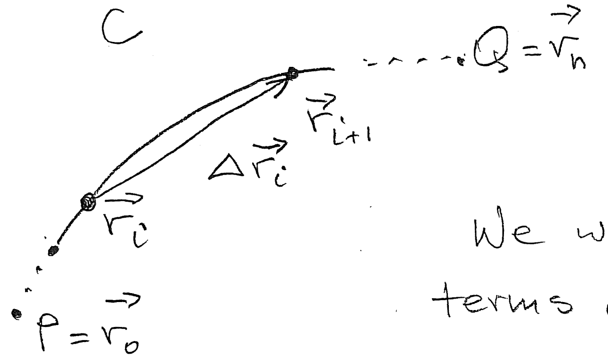
Thus $\vec{F}$ is not conservative.

So what vector fields are conservative? Gradient fields and only gradient fields. If

$$\vec{F}(x,y) = \nabla f(x,y)$$

for some $f(x,y)$, then $\vec{F}$ is conservative and vice-versa. Why? Let me give some justification.

Let $\vec{F}(x,y) = \nabla f(x,y)$ for some $f(x,y)$. Let C be an oriented smooth path from P to Q.



$$\int_C \vec{F}\, d\vec{r} = \lim_{\|\Delta\vec{r}\|\to 0} \sum_{i=0}^{n-1} \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i$$

We will express $\vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i$ in terms of f. Observe:

$$f(\vec{r}_{i+1}) - f(\vec{r}_i) \cong f_{\frac{\Delta\vec{r}_i}{\|\Delta\vec{r}_i\|}}(\vec{r}_i)\cdot\|\Delta\vec{r}_i\|$$

The directional derivative at $\vec{r}_i$ in the direction $\frac{\Delta\vec{r}_i}{\|\Delta\vec{r}_i\|}$, i.e. the rate of change

magnitude of the displacement in the direction $\frac{\Delta\vec{r}_i}{\|\Delta\vec{r}_i\|}$

But $f_{\frac{\Delta\vec{r}_i}{\|\Delta\vec{r}_i\|}}(\vec{r}_i) = \nabla f(\vec{r}_i)\cdot\frac{\Delta\vec{r}_i}{\|\Delta\vec{r}_i\|}$. Thus

$$f(\vec{r}_{i+1}) - f(\vec{r}_i) \cong \nabla f(\vec{r}_i)\cdot\Delta\vec{r}_i = \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i$$

Thus:

$$\sum_{i=0}^{n-1} \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i \cong \sum_{i=0}^{n-1}\left(f(\vec{r}_{i+1}) - f(\vec{r}_i)\right) =$$

$$= f(\vec{r}_1) - f(\vec{r}_0) + f(\vec{r}_2) - f(\vec{r}_1) + f(\vec{r}_3) - f(\vec{r}_2) + \dots + f(\vec{r}_n) - f(\vec{r}_{n-1}) =$$

$$= f(\vec{r}_n) - f(\vec{r}_0) = f(Q) - f(P)$$

Thus

$$\int_C \vec{F}\, d\vec{r} = f(Q) - f(P).$$

This is not a precise proof! Only justification.

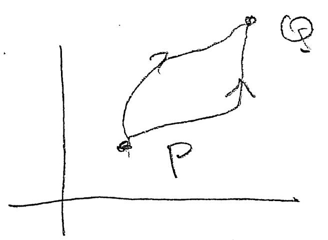
Th: A continuous vector field $\vec{F}$ defined in an open region of the xy-plane is path-independent if and only if $\vec{F}$ is a gradient field; that is:

$$\vec{F} = \nabla f$$

for some function f. In that case, for any piecewise smooth path C from P to Q we have:

$$\int_C \vec{F}\, d\vec{r} = \int_C \nabla f\, d\vec{r} = f(Q) - f(P).$$

The Fundamental Theorem for Line Integrals.



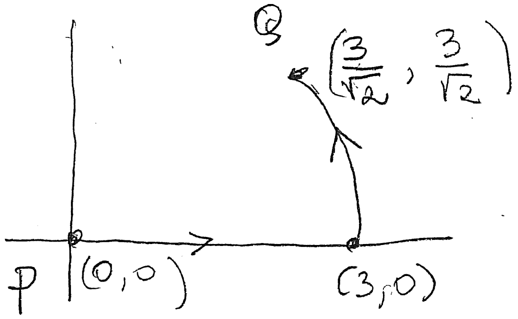
$$\int_P^Q \vec{F}\, d\vec{r} = \int_P^Q \nabla f\, d\vec{r} = f(Q) - f(P).$$

Remember the FTC from Calc I?

$$\int_a^b f'(t)\, dt = f(b) - f(a)$$

Def: If $\vec{F} = \nabla f$, then f is called a potential function of $\vec{F}$.

Ex: Let $\vec{F}(x,y) = x\vec{i} + y\vec{j}$, C be the path:



Find $\int_C \vec{F}\, d\vec{r}$.

We could parametrize both pieces: the segment and a piece of the circle of radius 3. But do we have to?

If $\vec{F} = \nabla f$, then $\int_C \vec{F} = f(Q) - f(P)$.

Is $\vec{F}$ a gradient field?

$$\vec{F}(x,y) = x\vec{i} + y\vec{j} \stackrel{?}{=} f_x \vec{i} + f_y \vec{j}$$

$f_x = x$, $f_y = y$. Possible.

$$f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$$

(Or $f(x,y) + C$ for any constant C.).

Hence:

$$\int_C \vec{F}\, d\vec{r} = \int_C \nabla f\, d\vec{r} = f(Q) - f(P) =$$

$$= \frac{1}{2}\left(\frac{3}{\sqrt{2}}\right)^2 + \frac{1}{2}\left(\frac{3}{\sqrt{2}}\right)^2 - 0 = \underline{\underline{\frac{9}{2}}}.$$