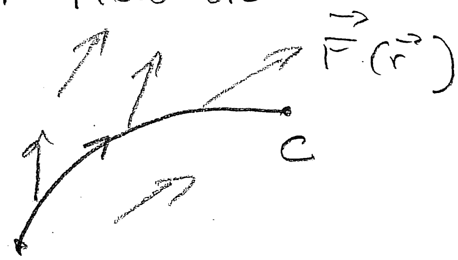
S10 Apr 20, 27

18.1 Line Integrals

Let a vector field $\vec{F}(\vec{r})$ and an oriented curve C on the xy-plane we given. "Oriented" means that we have a direction of motion on C. How do we define the line integral:

$$\int_C \vec{F}(\vec{r})\,d\vec{r} \quad ?$$

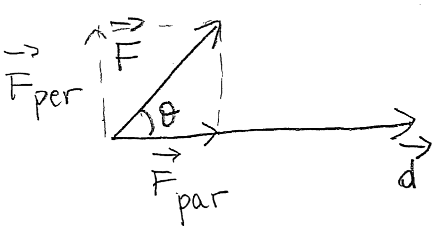


We define it in such a way that if $\vec{F}(\vec{r})$ is a force-field, then the integral represents the work done be the force as the body moves along C. What definition will capture that?

Recall the basic formula:

$$\text{Work} = \text{Force} \times \text{Distance}$$

which is valid provided an object moves along a directected segment in the exact direction of the force. What if the displacement and the force are not parallel:



$$\text{Work} = \|\vec{F}_{par}\| \cdot \|\vec{d}\|$$

$$\|\vec{F}_{per}\| = \|\vec{F}\| \cdot \cos\theta$$

The dot product.

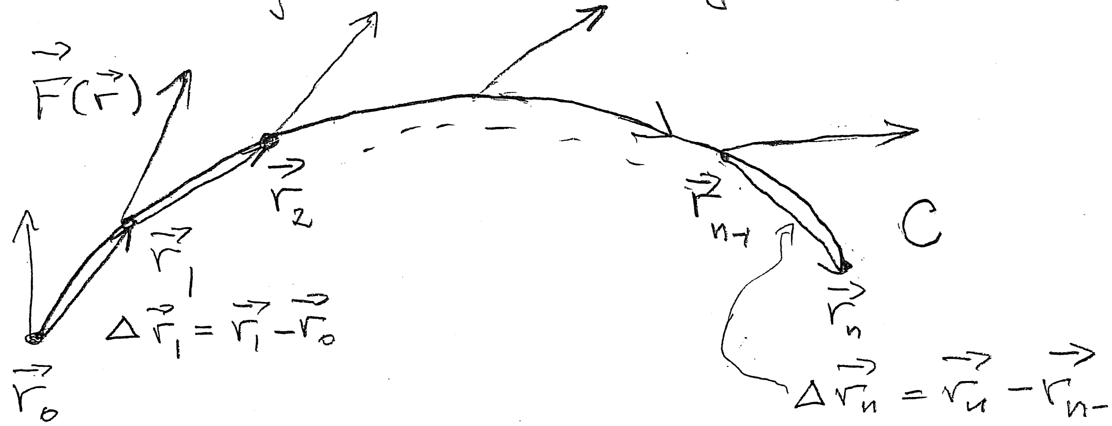
So: $\underline{\text{Work}} = \|\vec{F}\| \cdot \|\vec{d}\| \cdot \cos\theta = \underline{\vec{F} \cdot \vec{d}}$

Work is the dot product of the force and displacement.

Now we will define

$$\int_C \vec{F}(\vec{r})\, d\vec{r}$$

in such a way that the integral represent work.



We divide C into very small pieces by choosing points $\vec{r}_0, \vec{r}_1, \dots, \vec{r}_n$ along C. If C is smooth each piece is almost a segment so the displacement from $\vec{r}_{i-1}$ to $\vec{r}_i$ along C practically coincides with the displacement along the vector $\Delta\vec{r}_i = \vec{r}_i - \vec{r}_{i-1}$. $\vec{F}(\vec{r})$ changes along a small piece very little as well (if $\vec{F}(\vec{r})$ is continuous). Thus:

$$\text{Work over the } i\text{-th piece} \cong \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i$$

$$\text{Total work} \cong \sum_i \vec{F}(\vec{r}_i)\,\Delta\vec{r}_i$$

We get better and better approximation as $\|\Delta\vec{r}_i\| \to 0$:

$$\text{Total work} = \lim_{\|\Delta\vec{r}_i\|\to 0} \sum_i \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i \triangleq \int_C \vec{F}(\vec{r})\, d\vec{r}$$

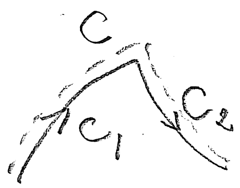
Under proper regularity assumptions about $\vec{F}$ and C (C smooth, $\vec{F}$ continuous) the limit exists and does not depend on the choice of $\vec{r}_i$. This limit is the line integral:

$$\int_C \vec{F}(\vec{r})\,d\vec{r} = \lim_{\|\Delta \vec{r}_i\| \to 0} \sum_i \vec{F}(\vec{r}_i)\cdot \Delta \vec{r}_i$$

Physically the integral represents work. Mathematically it represents "how much" the vector field $\vec{F}$ "goes in the direction" of C, "along C":

$\vec{F}\cdot\Delta\vec{r}$ $\qquad$ $\vec{F} \perp C \rightarrow \int_C \vec{F} = 0$

A few additional definitions and properties of the line integral:

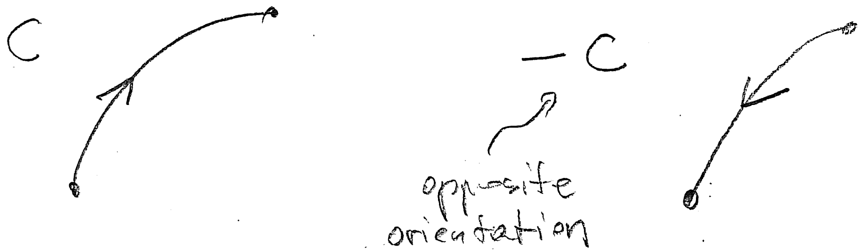


$C = C_1 + C_2$

Notation

The definition of $\int_C \vec{F}\,d\vec{r}$ can be extended to piecewise smooth path.

$$\int_C \vec{F}(\vec{r})\,d\vec{r} = \int_{C_1} \vec{F}(\vec{r})\,d\vec{r} + \int_{C_2} F(\vec{r})\,d\vec{r}$$

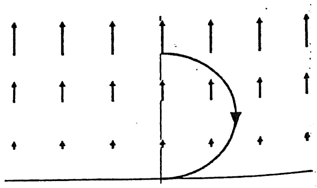


For a scalar constant λ, vector fields $\vec{F}$ and $\vec{G}$, and oriented curves C, C_1, and C_2

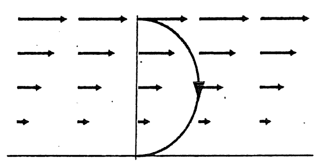
1. $\int_C \lambda\vec{F}\cdot d\vec{r} = \lambda\int_C \vec{F}\cdot d\vec{r}$.
2. $\int_C (\vec{F}+\vec{G})\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_C \vec{G}\cdot d\vec{r}$.
3. $\int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$.
4. $\int_{C_1+C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$.

Ex : Is $\int_C \vec{F}(\vec{r})d\vec{r}$ positive, negative or 0 ? C and $\vec{F}$ are depicted below.

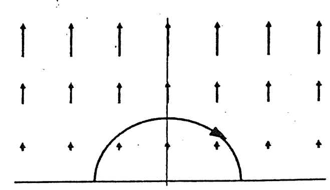
(a)



(b)



(c)



Negative. Positive. 0.

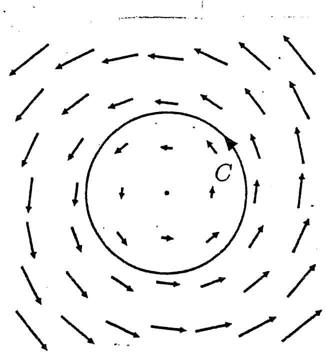
Def : If C is a closed oriented curve, then

$$\int_C \vec{F}(\vec{r})d\vec{r} = \oint_C \vec{F}(\vec{r})d\vec{r}$$

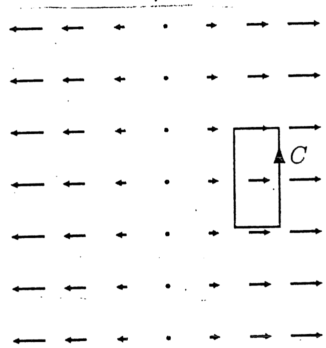
is called the circulation of $\vec{F}$ around C.

Ex : Find the sign of the circulation of the vector fields around the indicated paths.

(a)

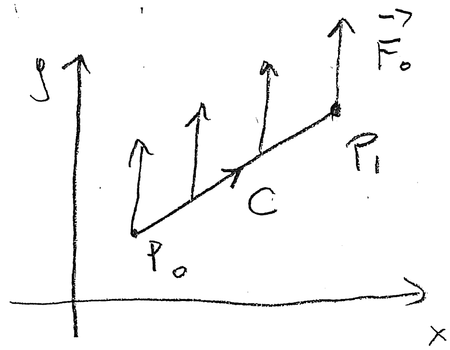


(b)



Positive. Zero.

<u>Remark</u>: Let $\vec{F}(\vec{r}) = \vec{F}_0$ be constant. Let C be an oriented straight line segment from P_0 to P_1.



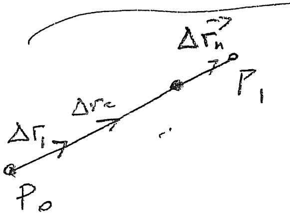
Then:

$$\int_C \vec{F}(\vec{r})\,d\vec{r} = \vec{F}_0 \cdot \overrightarrow{P_0P_1}$$

Follows easily from the definition of the line integral:

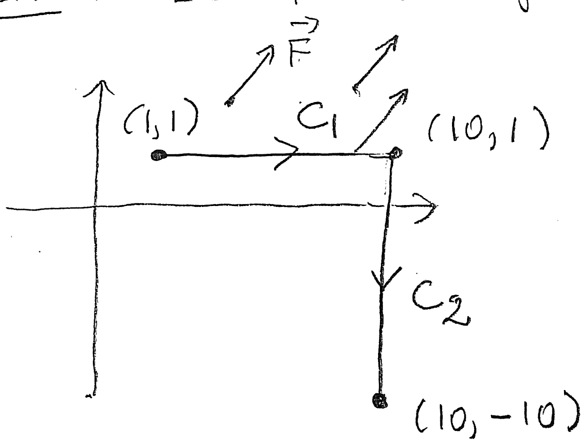
$$\int_C \vec{F}\,d\vec{r} = \lim_{\|\Delta\vec{r}_i\|\to 0} \sum_i \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i = \lim_{\|\Delta\vec{r}_i\|\to 0} \sum_i \vec{F}_0\cdot\Delta\vec{r}_i =$$

$$= \lim_{\|\Delta\vec{r}_i\|\to 0} \vec{F}_0\cdot\Big(\sum_i \Delta\vec{r}_i\Big) = \lim_{\|\Delta\vec{r}_i\|\to 0} \vec{F}_0\cdot\overrightarrow{P_0P_1} = \vec{F}_0\cdot\overrightarrow{P_0P_1}$$



$$\sum_i \Delta\vec{r}_i = \overrightarrow{P_0P_1}$$

<u>Ex</u>: Let $\vec{F} = \vec{i} + 2\vec{j}$, C be the following path:



$$C = C_1 + C_2$$

$$\int_C \vec{F}\,d\vec{r} = \int_{C_1} \vec{F}\,d\vec{r} + \int_{C_2} \vec{F}\,d\vec{r} =$$

$$= (\vec{i}+2\vec{j})\cdot(9\vec{i}) +$$

$$+ (\vec{i}+2\vec{j})\cdot(-11\vec{j}) =$$

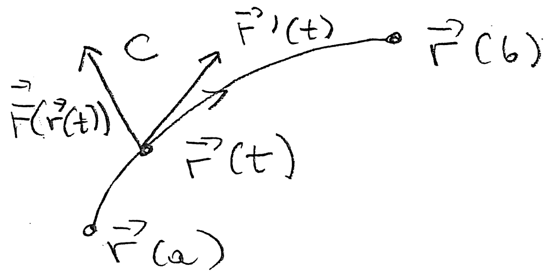
$$= 9 - 22 = \underline{-13}$$

In some simple cases we can find the line integral from the Remark or from the definition. What do we do in general? Parametrize C.

If $\vec{r}(t)$, for $a \le t \le b$, is a smooth parameterization of an oriented curve C and $\vec{F}$ is a vector field which is continuous on C, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt.$$

In words: To compute the line integral of $\vec{F}$ over C, take the dot product of $\vec{F}$ evaluated on C with the velocity vector, $\vec{r}\,'(t)$, of the parameterization of C, then integrate along the curve.



$$\int_C \vec{F}\, d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

dot product.

In terms of coordinates:

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}, \quad t \text{ in } [a, b],$$

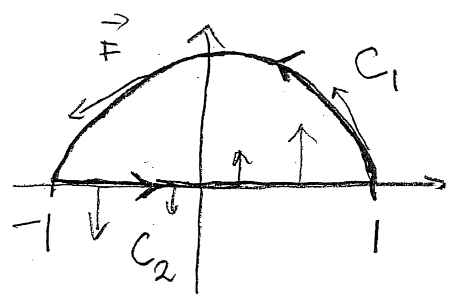
$$\vec{F}(\vec{r}(t)) = F_1(x(t), y(t))\vec{i} + F_2(x(t), y(t))\vec{j}$$

$$\int_C \vec{F}\, d\vec{r} = \int_a^b \left(F_1(x(t), y(t))\vec{i} + F_2(x(t), y(t))\vec{j}\right) \cdot \left(x'(t)\vec{i} + y'(t)\vec{j}\right) dt$$

Ex: Find $\int_C \vec{F}(\vec{r})\,d\vec{r}$ where:

$$\vec{F}(x,y) = -y\vec{i} + x\vec{j},$$

C is given by $C = C_1 + C_2$:



C_1 - the upper hemicircle of the unit circle oriented ccw.

C_2 - the segment $(-1,0),(1,0)$.

$$\int_C \vec{F}\,d\vec{r} = \int_{C_1} \vec{F}\,d\vec{r} + \int_{C_2} \vec{F}\,d\vec{r}$$

Note $\vec{F} \perp C_2$ at each point along C_2. Thus $\int_{C_2} \vec{F} = 0$.

We can check by parametrization:

C_2: $\quad x(t) = t, \quad y(t) = 0, \quad t$ in $[-1,1]$.

Thus:

$$\int_{C_2} \vec{F}(\vec{r})\,d\vec{r} = \int_{-1}^{1} F(x(t),y(t))] \cdot (x'(t)\vec{i} + y'(t)\vec{j})\,dt =$$

$$= \int_{-1}^{1} (t\vec{j}) \cdot (\vec{i})\,dt = \int_{-1}^{1} 0\,dt = 0.$$

Now C_1.

C_1: $x(t) = \cos(t)$, $y(t) = \sin(t)$, t in $[0, \pi]$

or, in vector form:

$$\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j}$$

$$\vec{r}'(t) = -\sin(t)\vec{i} + \cos(t)\vec{j}.$$

$$\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t)) = -\sin(t)\vec{i} + \cos(t)\vec{j}.$$

Thus:

$$\int_{C_1} \vec{F}(\vec{r})d\vec{r} = \int_0^{\pi} (-\sin(t)\vec{i} + \cos(t)\vec{j}) \bullet (-\sin(t)\vec{i} + \cos(t)\vec{j})\,dt =$$

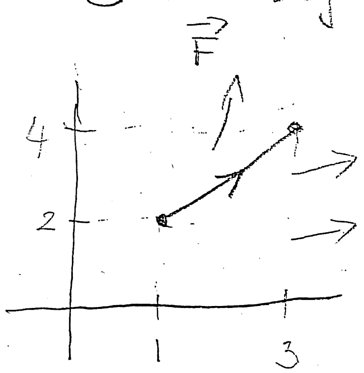
$$= \int_0^{\pi} \underbrace{(\sin^2(t) + \cos^2(t))}_{=1}\,dt = \pi.$$

Thus:

$$\int_C \vec{F}\,d\vec{r} = \pi$$

<u>Ex</u>. Find $\int_C \vec{F}$ where $\vec{F}(\vec{r}) = x^2\vec{i} + y^2\vec{j}$,

C the segment from $(1, 2)$ to $(3, 4)$.



Parametrize C:

$$\vec{w} = \overrightarrow{(1,2)(3,4)} = 2\vec{i} + 2\vec{j}$$

So we could take:

$x = 1 + 2t$, $y = 2 + 2t$, t in $[0, 1]$.

We can take instead a simpler parametrization:

C: $x(t) = 1 + t$, $y(t) = 2 + t$, t in $[0, 2]$.

We have:

$$\vec{r}(t) = (1+t)\vec{i} + (2+t)\vec{j},$$

$$\vec{r}\,'(t) = \vec{i} + \vec{j}$$

$$\vec{F}(\vec{r}(t)) = (1+t)^2\vec{i} + (2+t)^2\vec{j}.$$

So:

$$\int_C \vec{F}\,d\vec{r} = \int_0^2 \left((1+t)^2\vec{i} + (2+t)^2\vec{j}\right) \cdot (\vec{i} + \vec{j})\,dt =$$

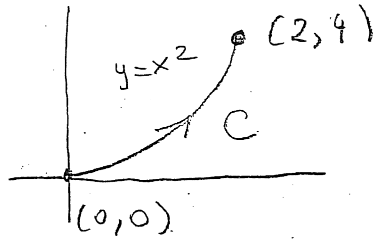
$$= \int_0^2 \left((1+t)^2 + (2+t)^2\right) dt = \int_0^2 (1 + 2t + t^2 + 4 + 4t + t^2)\,dt =$$

$$\int_0^2 (5 + 6t + 2t^2)\,dt = \left(5t + 3t^2 + \frac{2}{3}t^3\right)\Big|_0^2 = \frac{82}{3}.$$

Ex: Find $\int_C F(\vec{r})\,d\vec{r}$ where

$$\vec{F}(x,y) = -y\sin x\,\vec{i} + \cos x\,\vec{j},$$

C is the piece of the parabola $y = x^2$ from $(0,0)$ to $(2,4)$.



$C: \quad x(t)=t, \quad y(t)=t^2,$

t in $[0,2]$.

$$\vec{r}(t)=t\vec{i}+t^2\vec{j}, \quad \vec{r}\,'(t)=\vec{i}+2t\vec{j}$$

$$\vec{F}(\vec{r}(t))=-t^2\sin(t)\,\vec{i}+\cos(t)\,\vec{j}$$

$$\int_C \vec{F}\,d\vec{r}=\int_0^2\left(-t^2\sin(t)+2t\cos(t)\right)dt=$$

$-\sin(t)=(\cos(t))' \qquad 2t=(t^2)'$

$$=t^2\cos(t)\Big|_0^2=\underline{4\cos(2)}.$$

You can use your graphing calculator.