17.3 – 17.4 Vector Fields

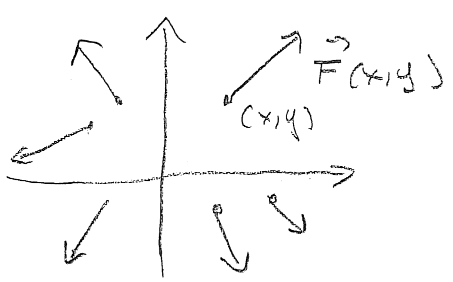
A vector field on the plane is a function which to each point (x,y) in some region prescribes a vector:

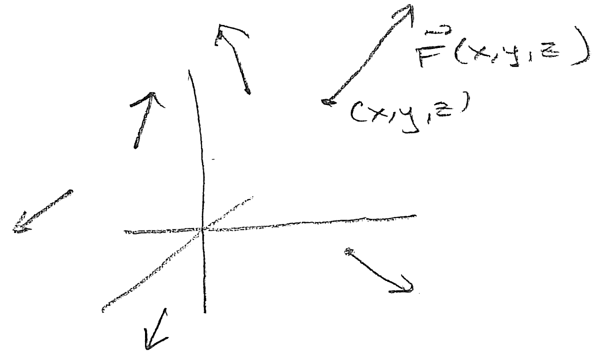
$$(x,y) \longrightarrow \vec{F}(x,y)$$

In 3D, it is a function which to each (x,y,z) prescribes a 3D vector $\vec{F}(x,y,z)$:

$$(x,y,z) \longrightarrow \vec{F}(x,y,z).$$

So:

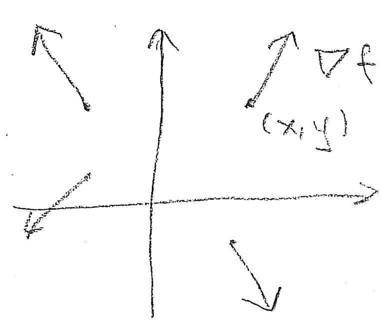




When we sketch vector fields, we usually scale length.
Vector fields are very important: force fields, current velocity fields etc.
We already know vector fields: if we have a function $f(x,y)$, then we have its <u>gradient vector field</u>:

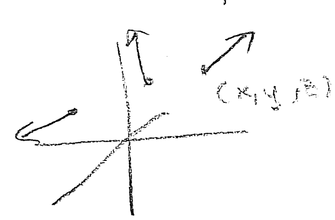
Given $f(x,y)$, we have

$$(x,y) \longrightarrow \nabla f(x,y)$$

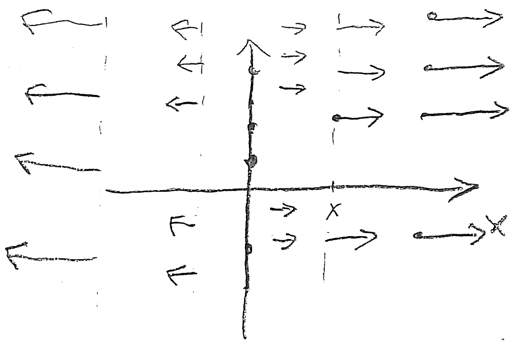


Given $f(x,y,z)$, we have

$$(x,y,z) \longrightarrow \nabla f(x,y,z)$$

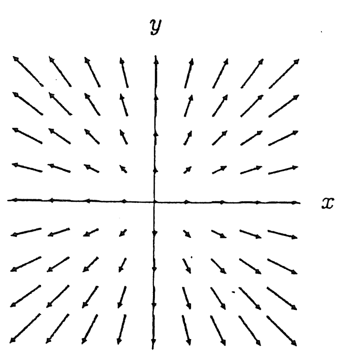


Ex: $f(x,y) = x^2$. Sketch the gradient vector field for f.



$$\nabla f(x,y) = 2x\vec{i}$$

Ex.: Sketch the vector field $\vec{F}(x,y) = 2x\vec{i} + 2y\vec{j}$.



Observe that $\vec{F}(x,y) = \text{grad}\, f(x,y)$, where $f(x,y) = x^2 + y^2$.

Denote $\vec{r} = x\vec{i} + y\vec{j}$. Then $\vec{F}(x,y)$ can be written as
$$\vec{F}(\vec{r}) = 2\vec{r}.$$

$\vec{F}(x,y) \parallel x\vec{i} + y\vec{j} = \vec{r}$ the position vector of (x,y)
$\vec{F}(x,y)$ at each point (x,y) point directly away from the origin. The magnitude increases as we move away from the origin. If possible, we write vector fields in terms of $\vec{r} = x\vec{i} + y\vec{j}$.

Def: A vector field $\vec{F}(x,y)$ or $(\vec{F}(x,y,z))$ is called a gradient vector field if for some $f(x,y)$ $(f(x,y,z))$:
$$\vec{F}(x,y) = \nabla f(x,y) \qquad (\vec{F}(x,y,z) = \nabla f(x,y,z)).$$

Perhaps every vector field is a gradient field?

Ex: $\vec{F}(x,y) = 2xy\,\vec{i} + xy\,\vec{j}$

Suppose that for some $f(x,y)$

$$\vec{F}(x,y) = \nabla f(x,y) = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}.$$

Then $f(x,y)$ has to be such that

$$f_x(x,y) = 2xy, \quad f_y(x,y) = xy$$

? $f(x,y) = x^2y + C(y)$

Doesn't seem possible.

$$f_y(x,y) = x^2 + C'(y) \quad \text{not } xy$$

There is a simple way to test if a given vector field

$$\vec{F}(x,y) = F_1(x,y)\,\vec{i} + F_2(x,y)\,\vec{j}$$

is a gradient field. If it is, then for some $f(x,y)$:

$$F_1(x,y) = f_x(x,y), \quad F_2(x,y) = f_y(x,y).$$

As $f_{xy}(x,y) = f_{yx}(x,y)$, we have then:

$$\frac{\partial F_1}{\partial y} = f_{xy} = f_{yx} = \frac{\partial F_2}{\partial x}.$$

So:

If $\dfrac{\partial F_1}{\partial y} \neq \dfrac{\partial F_2}{\partial x}$, then $\vec{F}$ is not a gradient field.

This is very important!

For $\vec{F}(x,y) = 2xy\vec{i} + xy\vec{j}$, we have:

$\frac{\partial}{\partial y}[2xy] = 2x$, $\frac{\partial}{\partial x}[xy] = y$

So $\frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$. Not a gradient field.