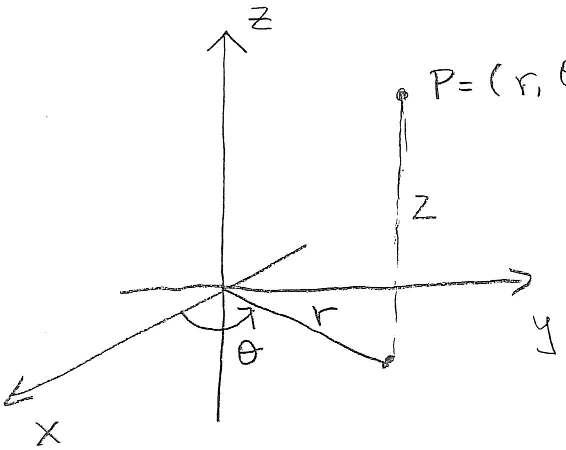
16.5 Triple Integrals in Spherical and Cylindrical Coords

Many solids in the xyz-space have much easier decription in cylindrical or spherical coordinates than in rectangular coordinates.

Cylindrical coordinates

Very simple: polar coordinates on the xy-plane, z stays the same.



$0 \leq r \leq +\infty$

$0 \leq \theta \leq 2\pi$

$-\infty < z \leq +\infty$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad , \quad x^2+y^2=r^2$$

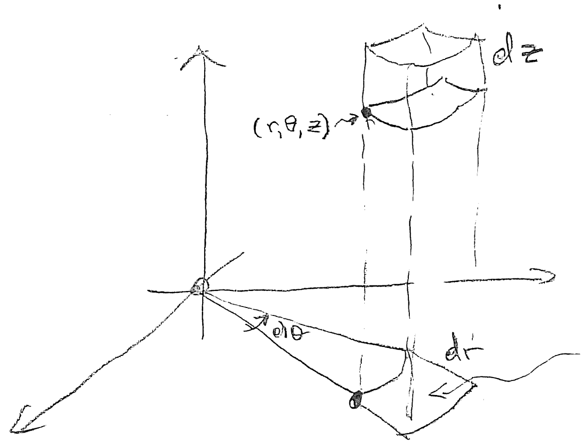
How do we convert a triple integral from rectangular to cylindrical coordinates? Easy.

$$\int_W f(x,y,z)\,dV = \int_W f(r\cos\theta, r\sin\theta, z)\, r\,\underbrace{d\theta\, dr\, dz}_{\text{or any other order.}}$$

In other words, the element of volume, dV, in cylindrical coordinates is:

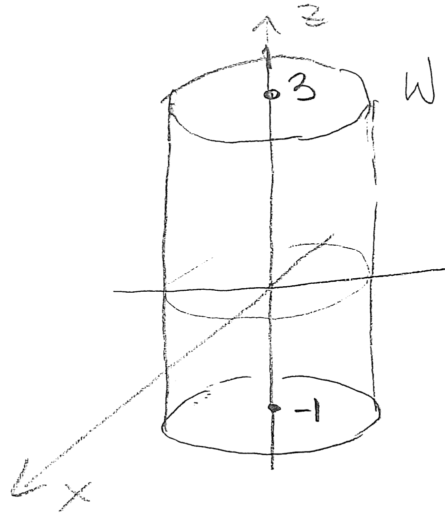
$$\underline{dV = r\,d\theta\,dr\,dz}$$ (or any permutation of $d\theta\,dz\,dr$)

Why? Let's look at the infinitesimal change in V corresponding to infinitesimal changes: $d\theta$, dr, dz:



$dV = r\,d\theta\,dr\,dz$

__Ex 1.__ Calculate $\int_W f(x,y,z)\,dV$ where $f(x,y,z) = \sin(x^2+y^2)$, W is the solid cylinder with height 4 and the base of radius 1 centered on the z-axis, on the plane $z=-1$.



In cylindrical coordinates:

$W: 0 \le r \le 1, 0 \le \theta \le 2\pi,$

$-1 \le z \le 3$

$f = \sin(r^2)$

The integral:

$$\int_0^1 \int_{-1}^3 \int_0^{2\pi} \sin(r^2)\, r\, d\theta\, dz\, dr =$$

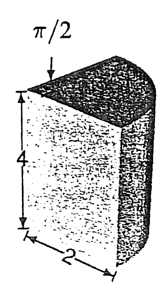
$$= \int_0^1 \int_{-1}^3 r\sin(r^2) \cdot 2\pi \, dz\, dr = \int_0^1 8\pi r \sin(r^2)\, dr =$$

$$= -4\pi \cos(r^2)\Big|_0^1 = \underline{-4\pi\cos(1) + 4\pi} \approx 5.78$$

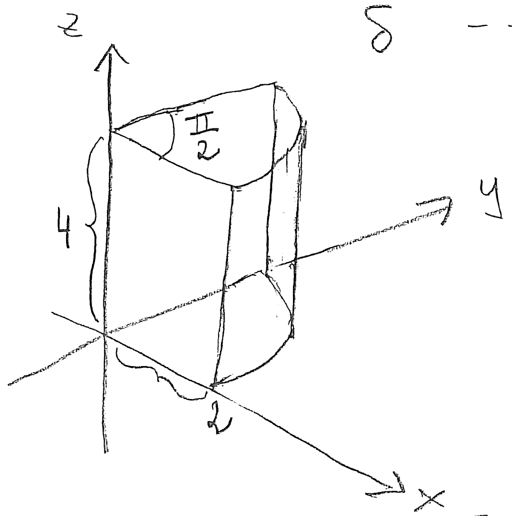
Ex 2:

For Problems 5–9, choose a set of coordinate axes, and then set up the three-variable integral in an appropriate coordinate system for integrating a density function δ over the given region.

6.



Let's choose an xyz-coordinate system so that the solid fits into the first octant. Then let's take the cylindrical coordinate system corresponding to this xyz-system as our solid is a wedge of a cylinder.



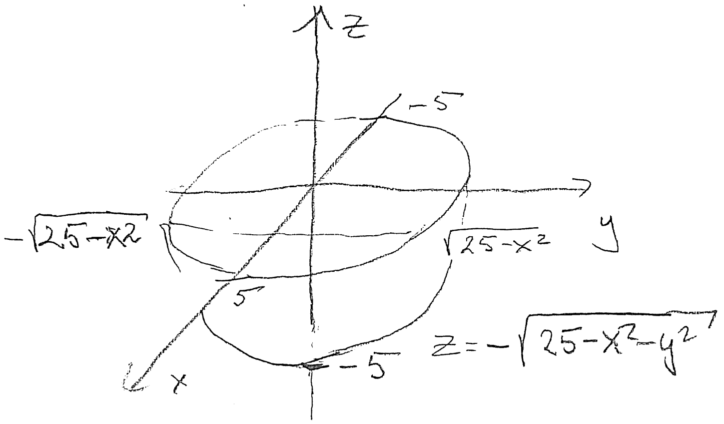
δ - the density function would have to be expressed in this coordinate system.

$$\int_W \delta(x,y,z)\,dV = \int_0^4 \int_0^{\frac{\pi}{2}} \int_0^2 \delta(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$$

Cylindrical solids have easy decriptions in the cylindrical coords. How about spherical solids?

Ex: Describe the solid of integration for the triple integral:

$$\int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \int_{-\sqrt{25-x^2-y^2}}^{0} f(x,y,z)\, dz\, dy\, dx$$



The bottom half of the sphere

$$x^2+y^2+z^2=25$$

It depends what $f(x,y,z)$ is but, in general, this integral is very hard to find. We need a coordinate system in which spheres have an easy description.

Spherical Coordinates

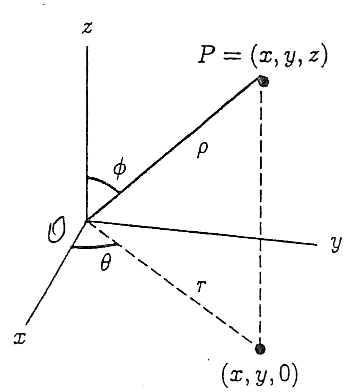


Figure 15.43: Spherical coordinates

Relation between Cartesian and Spherical Coordinates

Each point in 3-space is represented using $0 \le \rho < \infty, 0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$.

$(r = \rho \sin \Phi)$

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi.$$

Also, $\rho^2 = x^2+y^2+z^2$.

$P = (\rho, \theta, \Phi)$, $\rho = \text{dist}(P, O)$, θ the same as in Cylindrical coordinates, Φ the angle between the positive z-axis and OP.

dV in spherical coordinates:

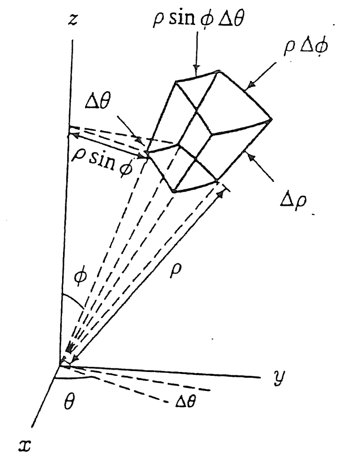


Figure 15.47: Volume element in spherical coordinates

When computing integrals in spherical coordinates, put $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$. Other orders of integration are also possible.

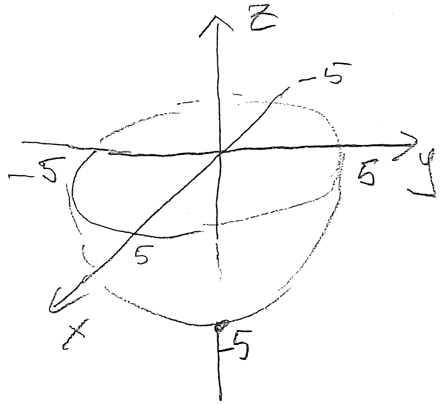
$$dV = \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

Ex: Evaluate

$$\int_W f(x,y,z)\, dV$$

where $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$, W is the bottom half of the sphere of radius 5 centered at the origin.

The same solid we just looked at. The integral practically impossible to do in rectangular coordinates.



$W: 0 \le \rho \le 5, \quad 0 \le \theta \le 2\pi$

$\frac{\pi}{2} \le \Phi \le \pi$

$$f(\rho, \theta, \Phi) = \frac{1}{\rho}$$

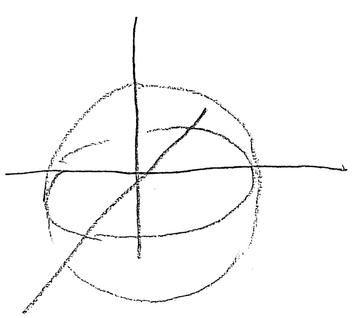
$$\int_W f\, dV = \int_0^5 \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{1}{\rho} \cdot \rho^2 \sin(\phi)\, d\phi\, d\theta\, d\rho =$$

$$= \int_0^5 \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \rho \sin\Phi\, d\Phi\, d\theta\, d\rho =$$

$$= \int_0^5 \int_0^{2\pi} \underbrace{\left(-\rho \cos\Phi \Big|_{\frac{\pi}{2}}^{\pi}\right)}_{= \rho} d\theta\, d\rho =$$

$$= \int_0^5 \int_0^{2\pi} \rho\, d\theta\, d\rho = \int_0^5 2\pi\rho\, d\rho = \pi\rho^2 \Big|_0^5 = \underline{25\pi}$$

<u>Ex</u>: Find the volume of the sphere of radius R.



$S: 0 \le \rho \le R, \ 0 \le \theta \le 2\pi, \ 0 \le \Phi \le \pi$

$$V = \int_0^R \int_0^{\pi} \int_0^{2\pi} \rho^2 \sin\Phi\, d\theta\, d\Phi\, d\rho =$$

$$= \int_0^R \int_0^\pi 2\pi \rho^2 \sin\Phi \, d\Phi \, d\rho =$$

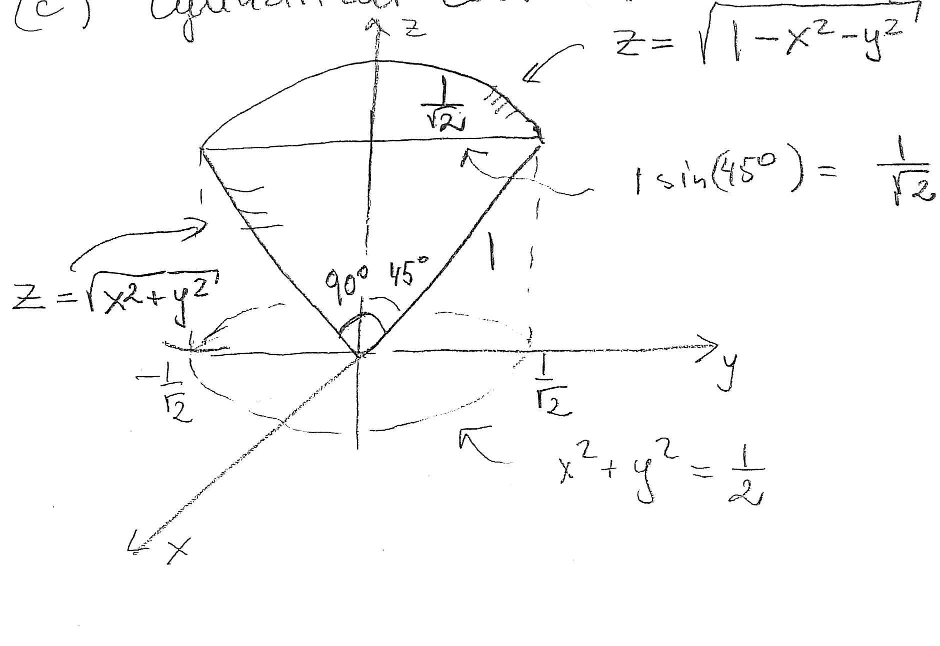
$$= \int_0^R 2\pi\rho^2 \underbrace{\left(-\cos\Phi \Big|_0^\pi\right)}_{2} d\rho = \int_0^R 4\pi \rho^2 d\rho =$$

$$= 4\pi \cdot \frac{1}{3}\rho^3 \Big|_0^R = 4\pi \cdot \frac{1}{3} R^3 = \underline{\underline{\frac{4}{3}\pi R^3}}$$

Ex: Let W be ~~t~~a cone with the angle 90° at the vertex topped by a sphere of radius 1. Write the iterated integral for

$$\int_W 1 \, dV$$

in (a) rectangular coords (b) spherical coords (c) cylindrical coords.



(a)

$$\int_W 1\, dV = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} 1 \, dz\, dy\, dx$$

(b)

$$\int_W 1 dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{1} 1\, \rho^2 \sin\Phi \, d\rho\, d\Phi\, d\theta$$

(c)

$$\int_W 1 dV = \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \int_r^{\sqrt{1-r^2}} 1\, r\, dz\, dr\, d\theta$$
