March 31
April 2

16.4 Double Integrals in Polar Coordinates

We are back to double integrals:

$$\int_R f(x,y)\,dA,$$

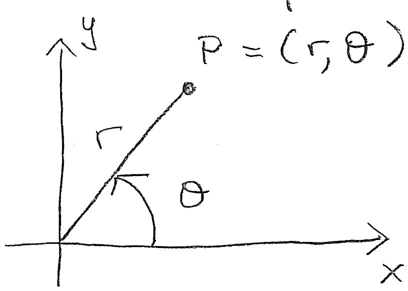
where $f(x,y)$ is a given function, R a given region on the xy-plane. As you saw, the key to setting up iterated integrals is the shape of R. So far we were using the rectangular coordinates to describe regions of integration. Many regions have a much easier description in polar coordinates. A quick refresher.

Polar Coordinates:

In polar coordinates every point P on the xy-plane is described by two coordinates:

$$P = (r, \theta)$$

where r is the distance of P from the origin O (called the pole), θ is the angle, in radians, between OP and the positive x-axis (called the polar axis) measured ccw:



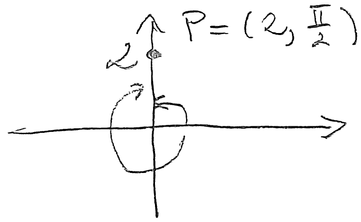
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad x^2+y^2 = r^2$$

Ex: Give polar coordinates of $P=(0,2)$ in rectangular coords.

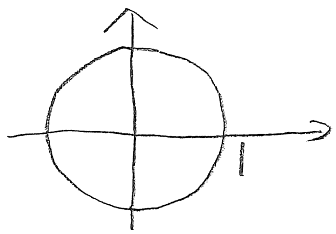
$$P = \left(2, \frac{\pi}{2}\right) \text{ also } P = \left(2, -\frac{3\pi}{2}\right),\ P = \left(2, \frac{5\pi}{2}\right) \ldots$$

The second polar coordinate is not unique:

If $P = (r, \theta)$, then $P = (r, \theta + 2\pi n)$, $n = 0, \pm 1, \pm 2, \ldots$



<u>Ex</u>: Find the equation of the unit circle in polar coordinates.



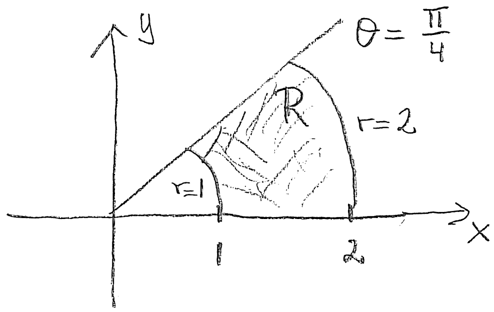
<u>In rectangular:</u>

$$x^2 + y^2 = 1$$

<u>In polar:</u>

$$r = 1$$

<u>Ex</u>: Describe the region R: $1 \le r \le 2$, $0 \le \theta \le \frac{\pi}{4}$.
(A "rectangle" in polar coordinates.).



R has a much more complicated decription in rectangular coordinates.

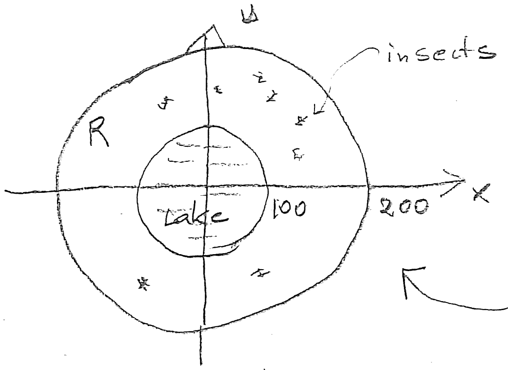
<u>Ex</u>: The density of insects in a circular region

$$100 \le \sqrt{x^2+y^2} \le 200$$

around a circular lake $0 \le \sqrt{x^2+y^2} \le 100$, x, y in meters, is given by

$$d(x,y) = \frac{1000}{\sqrt{x^2+y^2}} \quad \frac{\text{insects}}{m^2}$$

Find the total number of insects in the region.



$$\text{Total insects} = \int_R d(x,y)\,dA.$$

The density, d, depends only on the distance from the lake.

R is not easy to decribe in rectanglar coordinates.

In polar, though:

$$R: \ 100 \le r \le 200, \quad 0 \le \theta \le 2\pi$$

How to rewrite the double integral in polar coordinates?

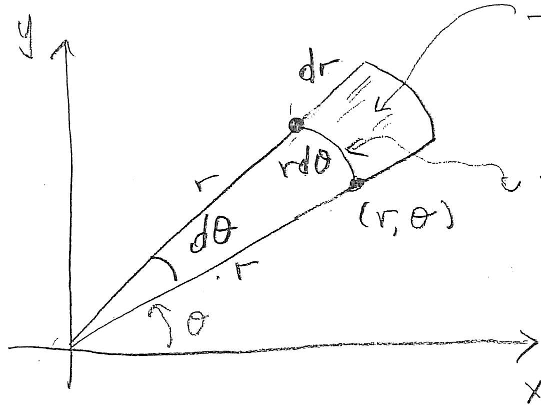
$$d(x,y) = \frac{1000}{\sqrt{x^2+y^2}} = \frac{1000}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} = \frac{1000}{r}$$

In other words:

$$\int_R d(x,y)\,dA = \int_R d(r\cos\theta, r\sin\theta)\,dA$$

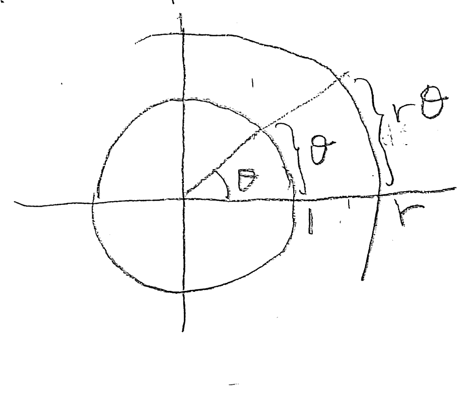
What is 'dA' in polar coordinates – the element of area?

'dA' is the infinitesimal change in area corresponding to infinitesimal changes dr, $d\theta$ in r and θ.



The change in area, dA, corresponding to the changes dr, $d\theta$.

$r\,d\theta$ by the definition of radian measure.

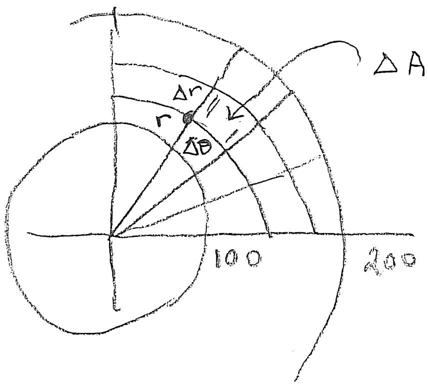


$$\underline{dA = r\,d\theta\,dr}$$

In polar coordinates the element of area depends on r :

$$dA = r\,d\theta\,dr = r\,dr\,d\theta$$

If you were calculating the integral over the region in polar coordinates, you would subdivide the region into small polar "rectangles" :



$\Delta A \cong r \cdot \Delta\theta \cdot \Delta r$

Insects in the little "rectangle" $\cong d(r,\theta) \cdot r \cdot \Delta\theta \cdot \Delta r$

$$\text{Total insects} = \lim_{\substack{\Delta\theta \to 0 \\ \Delta r \to 0}} \sum d(r,\theta) \cdot r\,\Delta\theta \cdot \Delta r$$

Thus :

$$\text{Total insects} = \int_R d\,dA = \int_R d(x,y)\,dx\,dy = \int_R d(r\cos\theta, r\sin\theta) \cdot r\,dr\,d\theta =$$

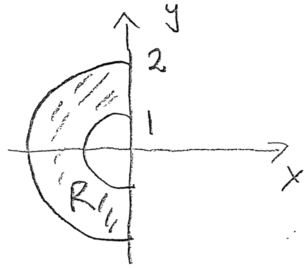
$$= \int_0^{2\pi} \int_{100}^{200} \frac{1000}{r} \cdot r\,dr\,d\theta = \int_0^{2\pi} 100{,}000\,d\theta = 2\pi \cdot 100{,}000 =$$

$$\cong 628{,}318 \text{ insects.}$$

In general :

$$\int_R f(x,y)\,dx\,dy = \int_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Ex: Write the integral $\int_R f \, dA$ as an iterated integral in polar coordinates, where R is

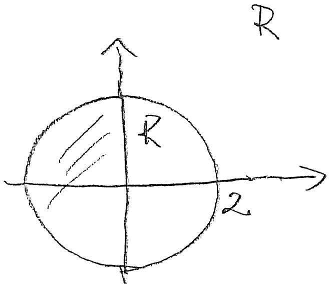


$$\int_R f \, dA = \int_1^2 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} f(r\cos\theta, r\sin\theta)\, r \, d\theta \, dr =$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_1^2 f(r\cos\theta, r\sin\theta)\, r \, dr \, d\theta .$$

Whichever is easier

Ex: $\int_R \sin(x^2+y^2)\, dA$, $R: \; x^2+y^2 \le 4$



Convert to polar:

$R: \; 0 \le r \le 2, \quad 0 \le \theta \le 2\pi$

$\sin(x^2+y^2) = \sin(r^2)$.

$$\int_R \sin(x^2+y^2)\, dA = \int_0^2 \int_0^{2\pi} \sin(r^2) . r . d\theta \, dr =$$

$$= \int_0^2 2\pi r \sin(r^2)\, dr = -\pi \cos(r^2) \Big|_0^2 =$$

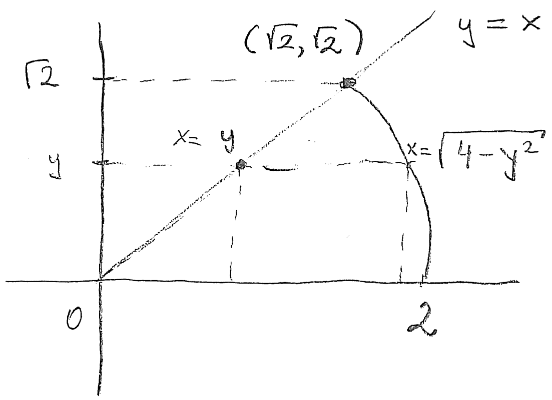
$$= \underline{-\pi \cos(4) + \pi}$$

Ex: Convert to polar coordinates. Evaluate.

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} (xy)\,dx\,dy$$

- Sketch the region R
- Describe the region in polar coordinates
- Set up an iterated integral in polar coordinates.

R: $0 \le y \le \sqrt{2}$, $y \le x \le \sqrt{4-y^2}$



$x = \sqrt{4-y^2}$

$x^2 + y^2 = 4$

Circle with radius 2.

$(\sqrt{2}, \sqrt{2})$ is on the circle.

So

$$R: \quad 0 \le r \le 2, \quad 0 \le \theta \le \frac{\pi}{4}$$

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} (xy)\,dx\,dy = \int_0^{\frac{\pi}{4}} \int_0^{2} r^2 \cos\theta \sin\theta \cdot r\,dr\,d\theta =$$

$$= \int_0^{\frac{\pi}{4}} 2\sin(2\theta)\,d\theta = -\cos(2\theta)\Big|_0^{\frac{\pi}{4}} = 0 - (-1) = \underline{1}$$

$$\int_0^2 r^3 \frac{1}{2} \sin(2\theta)\,dr = \frac{1}{2}\sin(2\theta)\left(\frac{1}{4} r^4 \Big|_0^2\right) = \frac{1}{2}\sin(2\theta)\cdot 4 =$$

$$= 2\sin(2\theta)$$