Ex: Consider the iterated integral:

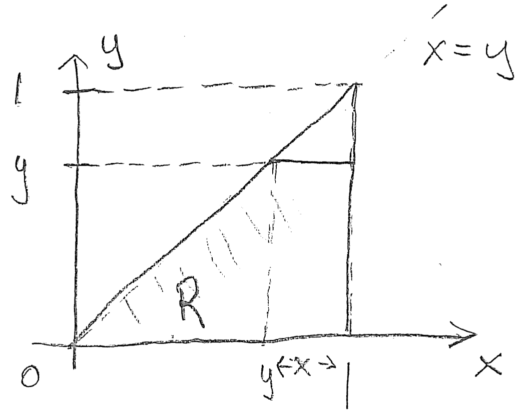
$$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$$

(a) Sketch the region of R for the corresponding double integral.

(b) Evaluate the integral.

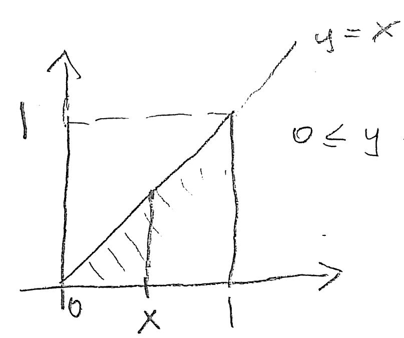
$0 \le y \le 1 \qquad y \le x \le 1$

R:



R: Triangle
$(0,0), (1,1), (1,0)$.

$$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy = \int_R e^{x^2}\, dA$$



$0 \le y \le x$

$\int_y^1 e^{x^2} dx$ cannot be found.

Change order of integration: $\int_R e^{x^2} dA = \int_0^1 \int_?^? e^{x^2} dy\, dx =$

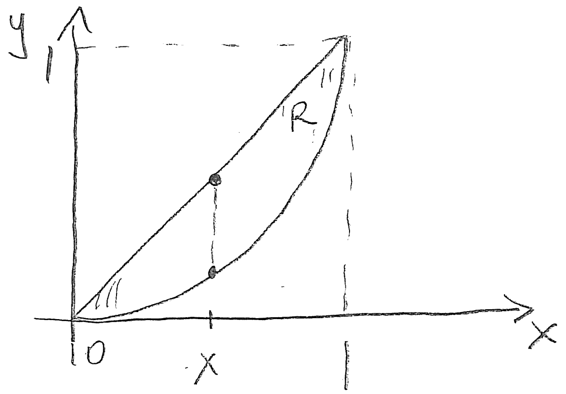
$$= \int_0^1 \left(\int_0^x e^{x^2} dy \right) dx = \int_0^1 \left(e^{x^2} y \Big|_{y=0}^{y=x} \right) dx = \int_0^1 x e^{x^2} dx =$$

$$= \frac{1}{2} e^{x^2} \Big|_0^1 = \underline{\frac{e}{2} - \frac{1}{2}}$$

Ex: Consider a metal plate R bounded by $y=x$ and $y=x^2$ with density of mass:

$$\delta(x,y) = 1 + xy \ \frac{\text{kg}}{\text{m}^2}$$

where x, y are in meters. Find the total mass of the plate.



$$M = \iint_R \delta(x,y)\,dA =$$

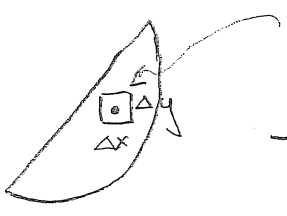
$$= \int_0^1 \left(\int_{x^2}^{x} (1+xy)\,dy \right) dx =$$

$$= \int_0^1 \left(x + \tfrac{1}{2}x^3 - x^2 - \tfrac{1}{2}x^5\right) dx = \left(\tfrac{1}{2}x^2 + \tfrac{1}{8}x^4 - \tfrac{1}{3}x^3 - \tfrac{1}{12}x^6\right)\Big|_0^1 = \frac{5}{24}\ \text{kg}$$

$$\int_{x^2}^{x} (1+xy)\,dy = \left(y + \tfrac{1}{2}xy^2\right)\Big|_{y=x^2}^{y=x} = \left(x + \tfrac{1}{2}x^3\right) - \left(x^2 + \tfrac{1}{2}x^5\right) =$$

$$= x + \tfrac{1}{2}x^3 - x^2 - \tfrac{1}{2}x^5$$

Why is M equal to the double integral?

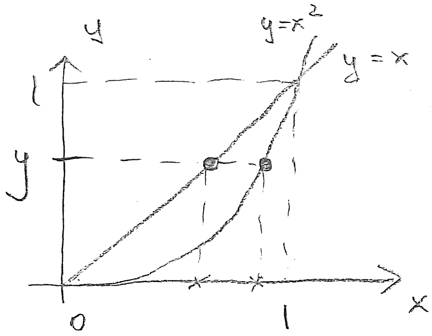


Mass of the Δx by Δy piece $\cong \delta(x,y)\cdot\Delta x\cdot\Delta y$

$$\text{Total mass} \cong \sum_{\substack{\text{over}\\ \text{all pieces}}} \delta(x,y)\,\Delta x\,\Delta y \xrightarrow[\substack{\Delta x \to 0\\ \Delta y \to 0}]{} \iint_R \delta(x,y)\,dA$$

$$M = \iint_R \delta(x,y)\,dA$$

How would the other iterated integral look?

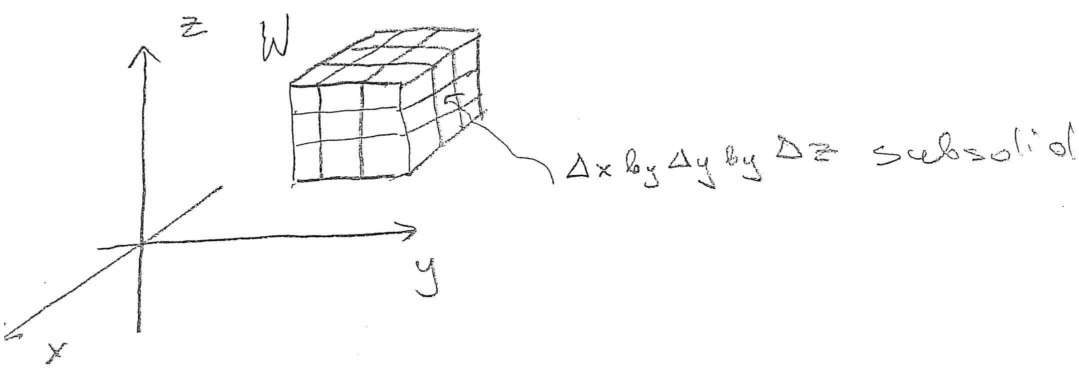


$$\int_0^1 \left(\int_y^{\sqrt{y}} \delta(x,y)\,dx \right) dy = \frac{5}{24}\text{ kg}$$

Triple Integrals

You can guess what they are. Let $f(x,y,z)$ be given and let a solid W in the xyz-space be given. For example, let W be a simple rectangular solid:

$$W: \quad a \le x \le b, \quad c \le y \le d, \quad p \le z \le q$$



We define the triple integral as:

$$\int_W f(x,y,z)\,dV = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \sum_{\substack{\text{all} \\ \text{subsolids}}} f(x_{i,j,k}, y_{i,j,k}, z_{i,j,k})\,\Delta x\,\Delta y\,\Delta z$$

The integral is the limit of Riemann sums obtained by dividing W into small subsolids Δx by Δy by Δz, choosing a point $(x_{i,j,k}, y_{i,j,k}, z_{i,j,k})$ from each subsolid and multiplying the value of f at that point by the volume of the subsolid.

Philosophically, $\int_W f(x,y,z)\,dV$ is "the value of f times the volume of W", but if f changes over W, we have to divide W into small pieces etc....

Clearly from the definition:

$$\int_W 1\,dV = \text{volume}(W).$$

Similarly, as

$$\int_R 1\,dA = \text{Area}(R).$$

Also similarly as for double integrals, we can find triple integrals via iterated integrals:

$$\int_W f(x,y,z)\,dV = \int_p^q \left(\int_c^d \left(\int_a^b f(x,y,z)\,dx \right) dy \right) dz$$

Also for non-rectangular solids.

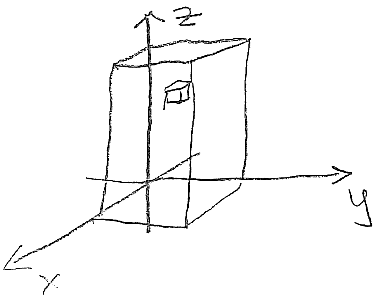
Ex: Consider a metal solid W in the xyz-space

W: $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 2$

where x, y, z are in centimeters. The density of mass in W is given by:

$$\delta(x,y,z) = (x+y+z+1)\ \frac{g}{cm^3}$$

Find the total mass of W.



If the density was constant, say

$\delta = 3\ g/cm^3$

what the total mass would be?

$M = \text{Vol}(W) \cdot \delta = 2\,cm^3 \cdot 3\ g/cm^3 = 6g.$

But the density is not constant so we have to divide the solid into small subsolids over which $\delta(x,y,z)$ change only a little, estimate the mass of each subsolid:

a point from the subsolid

$$\text{Mass of subsolid} \cong \delta(x,y,z) \cdot \Delta x \Delta y \Delta z$$

$$M = \text{Total mass} \cong \sum_{\substack{\text{over} \\ \text{all subsolids}}} \delta(x,y,z)\, \Delta x\, \Delta y\, \Delta z$$

In other words, we have to take a triple integral:

$$M = \int_W \delta(x,y,z)\, dV.$$

We evaluate the integral using iterated integrals:

$$M = \int_0^2 \int_0^1 \underbrace{\int_0^1 (x+y+z+1)\,dx}\,dy\,dz =$$

$$= \int_0^2 \int_0^1 \left(\left(\tfrac{1}{2}x^2 + yx + zx + x\right)\Big|_{x=0}^{x=1} \right) dy\,dz =$$

$$= \int_0^2 \int_0^1 \left(\tfrac{1}{2} + y + z + 1\right) dy\,dz = \int_0^2 \underbrace{\int_0^1 \left(\tfrac{3}{2} + y + z\right) dy}\,dz =$$

$$= \int_0^2 \left(\left(\tfrac{3}{2}y + \tfrac{1}{2}y^2 + zy\right)\Big|_{y=0}^{y=1} \right) dz =$$

$$= \int_0^2 \left(\tfrac{3}{2} + \tfrac{1}{2} + z\right) dz = \left(2z + \tfrac{1}{2}z^2\right)\Big|_0^2 = \underline{\underline{6\ g}}$$

After calculating each integral, one variable disappears.

Iterated integrals work for non-rectangular solids. You have to be careful about the limits.

Ex: Let $f(x,y,z) = x+y$. Find

$$\int_W f \, dV$$

where W is the solid bounded by the xy-plane, yz-plane, xz-plane, and the plane

$$P: \quad \frac{x}{3} + \frac{y}{2} + \frac{z}{6} = 1.$$

$z=0$:

$\frac{x}{3} + \frac{y}{2} = 1 \rightarrow y = -\frac{2}{3}x + 2$

$z = 6 - 2x - 3y$

Let's sketch the solid so we can set up an iterated integral.

Normal to P:

$$\vec{n}_P = \langle \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{6} \rangle$$

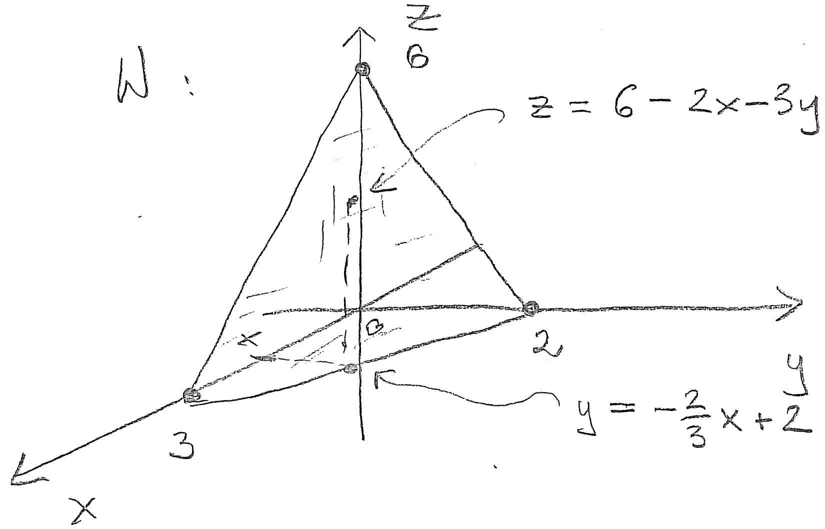
P intersects the

x-axis at $(3,0,0)$

y-axis at $(0,2,0)$

z-axis at $(0,0,6)$

W:



W is the pyramid in the first octant.

$$\int_W f \, dV = \int_0^3 \int_0^{-\frac{2}{3}x+2} \int_0^{6-2x-3y} (x+y) \, dz \, dy \, dx$$

$$\int_0^3 \Big(\int_0^{-\frac{2}{3}x+2} \Big(\underbrace{\int_0^{6-2x-3y} (x+y)\, dz}\Big) dy \Big) dx =$$

$$= \int_0^3 \Big(\int_0^{-\frac{2}{3}x+2} (x+y)(6-2x-3y)\, dy \Big)\, dx =$$

$$(x+y)(6-2x-3y) = 6x-2x^2-3xy+6y-2xy-3y^2 =$$

$$= 6x-2x^2-3y^2+6y-5xy$$

$$= \int_0^3 \Big(\underbrace{\int_0^{-\frac{2}{3}x+2} (6x-2x^2-3y^2+6y-5xy)\, dy}\Big)\, dx =$$

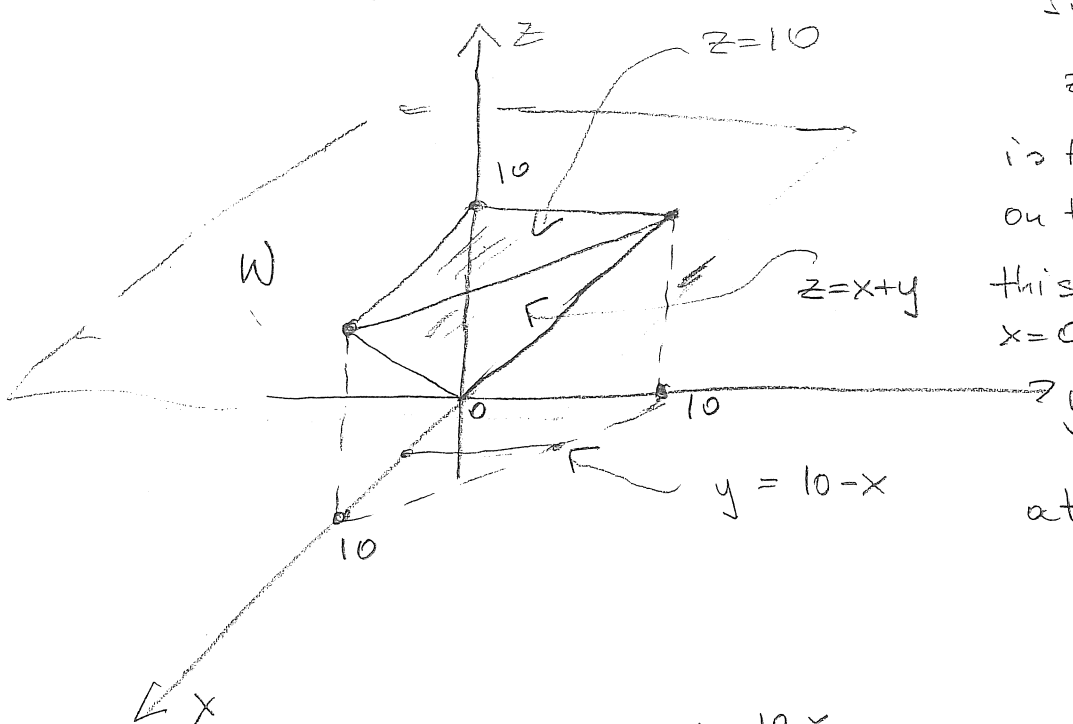
$$= \int_0^3 \underbrace{\Big(\Big(6xy-2x^2y-y^3+3y^2-\frac{5}{2}xy^2 \Big) \Big|_{y=0}^{y=-\frac{2}{3}x+2} \Big)}\, dx =$$

$$6x\left(-\frac{2}{3}x+2\right) - 2x^2\left(-\frac{2}{3}x+2\right) - \left(-\frac{2}{3}x+2\right)^3 + 3\left(-\frac{2}{3}x+2\right)^2 -$$

$$-\frac{5}{2}x\left(-\frac{2}{3}x+2\right)^2 = \underline{\frac{14}{27}x^3 - \frac{8}{3}x^2 + 2x + 4}$$

$$= \int_0^3 \left(\frac{14}{27}x^3 - \frac{8}{3}x^2 + 2x + 4 \right) dx = \underline{\underline{\frac{15}{2}}}$$

Ex: Find the volume of the solid bounded by $z = x+y$, $z = 10$, and the planes $x=0$; $y=0$.



Intersection of $z = x+y$ with $z=10$ is the $10 = x+y$ line on the $z=10$ plane. this line intersects $x=0$ plane at $(0, 10, 10)$ and $y=0$ plane at $(10, 0, 10)$.

$$\mathrm{Vol}(W) = \int_W 1\, dV = \int_0^{10}\int_0^{10-x}\int_{x+y}^{10} 1 \; dz\, dy\, dx =$$

$$= \int_0^{10}\int_0^{10-x} \left(z \Big|_{z=x+y}^{z=10} \right) dy\, dx =$$

$$= \int_0^{10} \left(\int_0^{10-x} (10-x-y)\, dy \right) dx = \int_0^{10} \left(\left(10y - xy - \tfrac{1}{2}y^2\right) \Big|_{y=0}^{y=10-x} \right) dx =$$

$$= \int_0^{10} \left(10(10-x) - x(10-x) - \tfrac{1}{2}(10-x)^2 \right) dx = \int_0^{10} \left(50 - 10x + \tfrac{1}{2}x^2\right) dx = \frac{500}{3}$$

$$(10-x)(10-x-5+\tfrac{1}{2}x) = (10-x)(5-\tfrac{1}{2}x) =$$
$$= 50 - 5x - 5x + \tfrac{1}{2}x^2 = 50 - 10x + \tfrac{1}{2}x^2$$

Ex: Sketch the solid of integration W corresponding to the iterated integral:

$$\int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} f(x,y,z)\,dx\,dy\,dz$$

W

upper half of the unit ball