Calculating Double Integrals Using Iterated Integrals

Let R be a rectangle $a \le x \le b$, $c \le y \le d$. Let $f(x,y)$ be continuous on R. Then

y fixed　　　　x fixed

$$\int_R f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

interated integrals

Ex: $f(x,y) = x^2 y$, $R: 0 \le x \le 1,\ 0 \le y \le 2.$

Evaluate $\int_R f(x,y)\,dA$.

$$\int_R (x^2 y)\,dA = \int_0^2 \int_0^1 (x^2 y)\,dx\,dy = \int_0^1 \int_0^2 (x^2 y)\,dy\,dx =$$

Integrate with respect to x first

Integrate with respect to y first. Let's do this one.

$$= \int_0^1 2x^2\,dx = \frac{2}{3}x^3 \Big|_0^1 = \underline{\underline{\frac{2}{3}}}$$

Inside integral:

$$\int_0^2 (x^2 y)\,dy = \left(x^2 \frac{1}{2} y^2\right) \Big|_0^2 = 2x^2$$

constant　　y=2　　y=0

We could use the other iterated integral. For rectangular regions it usually doesn't matter.

$$\int_0^2 \int_0^1 (x^2 y)\,dx\,dy = \int_0^2 \frac{1}{3} y\,dy = \frac{1}{6} y^2 \Big|_0^2 = \frac{4}{6} = \frac{2}{3}$$

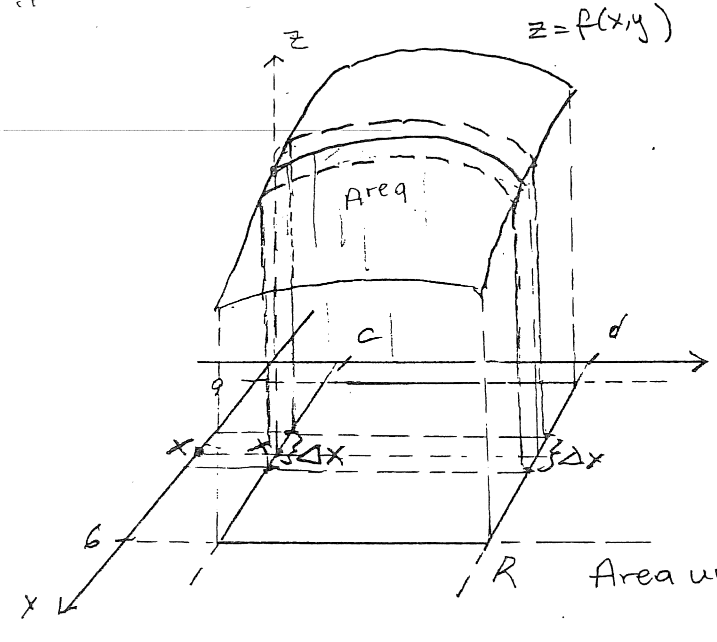
limits for x

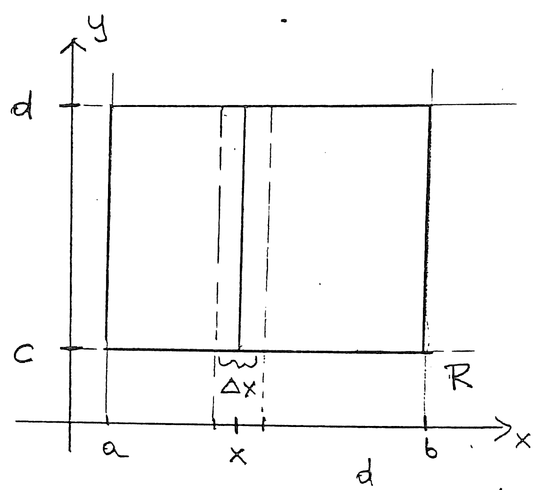
limits for y

$$\int_0^1 (x^2 y)\,dx = \frac{1}{3} x^3 y \Big|_{x=0}^{x=1} = \frac{1}{3} y$$

constant

Why do interated integrals work? We will justify why in the case when $f(x,y) \geq 0$ over R and R is a rectangle; In that case, $\int_R f\,dA$ is the volume under the graph. In the definition of the double integral, we calculated the volume by dividing into subrectangles and thin vertical pillars. The volume can be computed by dividing into slabs:





Area under the cross-section $f(x,y) = \int_c^d f(x,y)\,dy$

With x - fixed:

Volume above the x-slab $\cong$ Area $\cdot\, \Delta x \cong \left(\int_c^d f(x,y)\,dy\right) \cdot \Delta x$

Total volume $= \lim\limits_{\Delta x \to 0} \sum\limits_{\substack{\text{over} \\ x\text{-slabs}}} \left(\int_c^d f(x,y)\,dy\right) \cdot \Delta x = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$