16.1 The Double Integral of a Function of Two Variables

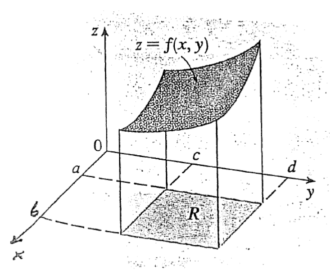
Let's have a function of two variables, f(x,y), over a region R contained in its domain. We want to define the double integral of the function f over R denoted as:

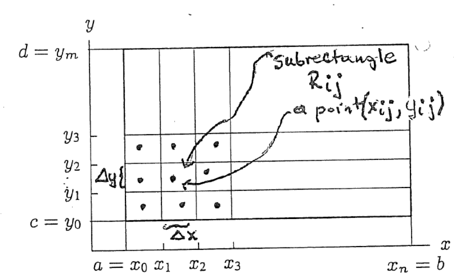
$$\int_R f(x,y)\, dA .$$

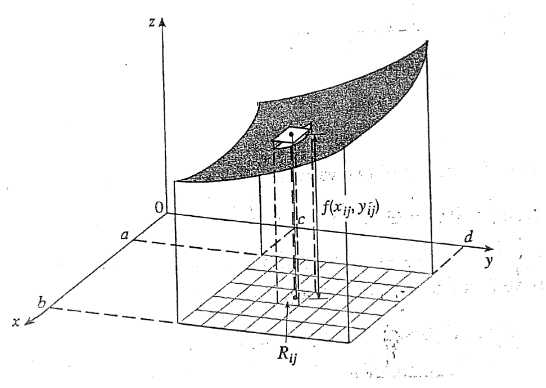
How is such integral defined? In terms of Riemann sums. Assume for simplicity that R is a rectangle:

$$R: \quad a \le x \le b \quad , \quad c \le y \le d .$$

(The definition is the same for an arbitrary region R.) The pictures should refresh your memory about the definition of Riemann sums and the double integral:







$$\int_R f(x,y)\,dx\,dy = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \underbrace{\sum f(x_{ij}, y_{ij})\,\Delta x\,\Delta y}_{\text{a Riemann sum}}$$

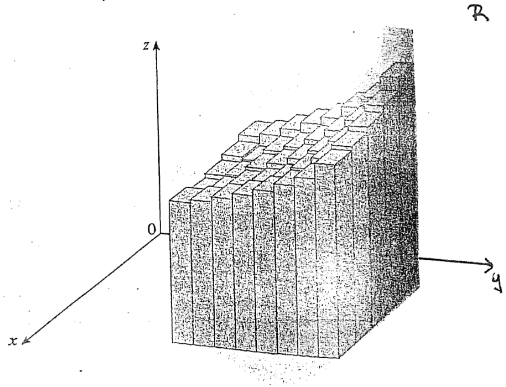
Also denoted:

$$\int_R f(x,y)\,dA = \int_R f(x,y)\,dx\,dy = \int_R f(x,y)\,dy\,dx.$$

The element of area.

Philosophically: $\int_R f(x,y)\,dA$ is "the value of the function f" times the area of R. But the value of f changes over R, so you have to subdivide R etc. etc.

If $f(x,y) \geq 0$ in R, then $\int_R f(x,y)\,dx\,dy$ is the volume under the graph:



Riemann sums approximate the volume. The approximation gets better and better as $\Delta x \to 0$, $\Delta y \to 0$.

Riemann sums are not very handy for evaluating double integrals. Iterated integrals are.

The double integral of a non-negative function f(x,y) is the volume under the graph of the function. From the definition of the integral in terms of Riemann sums, we easily see that if a function f(x,y) changes sign on R, the integral is the sum of volumes: the volumes above the xy-plane come into the sum with the plus sign, the volumes under the xy-plane with the minus sign. Hence, the integral of the function depicted in blue on the picture below over the rectangle in orange seems to be 0: